

AMES GRANT

IN-34-CR

19392  
P-38

Progress report on the NASA-Ames grant no. NCC2-186.

**NAVIER-STOKES COMPUTATION OF COMPRESSIBLE TURBULENT  
FLOWS WITH A SECOND ORDER CLOSURE.**

Time period: 05/01/90 - 04/30/91.

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Objective.

The objective of the present part of the project was the development of a complete second order closure for wall bounded flows including all components of the dissipation rate tensor and a numerical solution procedure for the resulting system of equations. The main topics of the present grant were the closure of the pressure correlations and the viscous destruction terms in the dissipation rate equations and the numerical solution scheme based on a block-tridiagonal solver for the nine equations required for the prediction of plane or axisymmetric flows.

(NASA-CR-188451) NAVIER-STOKES COMPUTATION  
OF COMPRESSIBLE TURBULENT FLOWS WITH A  
SECOND ORDER CLOSURE Progress Report, 1 May  
1990 - 30 Apr. 1991 (California Univ.)  
38 p

N91-24543

Unclass  
0019392

CSCL 20D G3/34



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## 1.0 Full second order closure for wall bounded flows.

### 1.1 Introduction.

A full second order closure for wall bounded shear flows is developed, which includes the Reynolds stress equations and the equations for all relevant components of the dissipation rate tensor. Incompressible and compressible plane flows are considered, but this report is only concerned with incompressible flows.

There are several reasons for the development of a complete second order closure. It can be shown that the anisotropy of stress and dissipation rate tensors approaches the same limit at the wall, but the derivative of the anisotropy of the dissipation rate is twice the derivative of the stress anisotropy at the wall. Another reason is the possibility of constructing appropriate time scales in the near wall region. The standard second order closure incorporates the transport equation for the trace of the dissipation rate tensor and relates the components of the tensor via local relations to the trace. The time scale for the destruction of the trace is usually modelled using the time scale

$$\tau_D^{-1} = \frac{\hat{\epsilon}}{k}$$

where the modified dissipation rate  $\hat{\epsilon}$  is defined by

$$\hat{\epsilon} \equiv \epsilon - 2\nu \left( \frac{\partial \sqrt{k}}{\partial y} \right)^2$$

with  $y$  denoting the coordinate normal to the wall. This is an acceptable model in the region close to the wall if and only if the dissipation rate is a nondecreasing function of the distance from the wall, because the kinetic energy is of order  $O(k) = y^2$  near the wall which implies that the second term in the modified dissipation rate is constant. Direct simulations of boundary layers and channel flows, however, have shown (see Mansour et al., 1988) that the dissipation rate is a rapidly decreasing function of the wall distance in the viscous sublayer and the destruction model using the time scale  $\tau_D$  becomes thus a production term in the near wall region. This is clearly a violation of realizability for the destruction model. It follows that this type of closure model does not represent properly the production of dissipation rate near the wall and the sign reversal of the viscous destruction model must make up for this deficit in production. The motivation for the modified time scale is the fact that the time scale

$$\tau \equiv \frac{k}{\epsilon}$$

goes to zero as the wall is approached. It will be shown below that the full dissipation rate tensor allows a realizable and tensorially invariant construction of a time scale or a time scale tensor that reaches a finite and nonzero limit value at the wall.

A second point that sets the near wall region apart from the high Reynolds number regime of the boundary layer is the growth of the pressure correlations with distance from the wall. It will be shown below that the usual split of the pressure correlations into pressure transport and pressure rate of strain correlations is not appropriate near the wall, because the split correlations grow with different rates and the Taylor series for the original correlation involving the fluctuating pressure gradient can be expressed locally up to second order in terms of velocity correlations.

## 1.2 Exact equations for the dissipation rate tensor.

Incompressible flows are considered first. Standard manipulations lead to the transport equations for the dissipation rate tensor in a Cartesian coordinate system defined by

$$\epsilon_{\alpha\beta} \equiv 2\nu\partial_\gamma v'_\alpha \partial_\gamma v'_\beta \quad (1)$$

with expectation denoted by  $\langle \epsilon_{\alpha\beta} \rangle$ . The equations can be given in the form

$$(\partial_t + \langle v_\gamma \rangle \partial_\gamma) \langle \epsilon_{\alpha\beta} \rangle = \partial_\gamma [\nu \partial_\gamma \langle \epsilon_{\alpha\beta} \rangle - \langle v'_\gamma \epsilon_{\alpha\beta} \rangle] + S^1_{\alpha\beta} + S^2_{\alpha\beta} + S^3_{\alpha\beta} + S^4_{\alpha\beta} + S^5_{\alpha\beta} - D_{\alpha\beta} \quad (2)$$

The various source terms are defined as follows.

$$S^1_{\alpha\beta} \equiv -\langle \epsilon_{\beta\gamma} \rangle \partial_\gamma \langle v_\alpha \rangle - \langle \epsilon_{\alpha\gamma} \rangle \partial_\gamma \langle v_\beta \rangle \quad (3)$$

$$S^2_{\alpha\beta} \equiv -\langle \epsilon_{\alpha\beta}^{\gamma\delta} \rangle (\partial_\gamma \langle v_\delta \rangle + \partial_\delta \langle v_\gamma \rangle) \quad (4)$$

$$S^3_{\alpha\beta} \equiv -\langle \epsilon_{\beta\gamma} \partial_\gamma v'_\alpha \rangle - \langle \epsilon_{\alpha\gamma} \partial_\gamma v'_\beta \rangle \quad (5)$$

$$S^4_{\alpha\beta} \equiv -\frac{2\nu}{\rho} (\langle \partial_{\alpha\gamma}^2 p' \partial_\gamma v'_\beta \rangle + \langle \partial_{\beta\gamma}^2 p' \partial_\gamma v'_\alpha \rangle) \quad (6)$$

$$S^5_{\alpha\beta} \equiv -2\nu (\langle v'_\delta \partial_\gamma v'_\beta \rangle \partial_{\gamma\delta}^2 \langle v_\alpha \rangle + \langle v'_\delta \partial_\gamma v'_\alpha \rangle \partial_{\gamma\delta}^2 \langle v_\beta \rangle) \quad (7)$$

$$D_{\alpha\beta} \equiv 4\nu^2 \langle \partial_{\delta\gamma}^2 v'_\alpha \partial_{\delta\gamma}^2 v'_\beta \rangle \quad (8)$$

Furthermore is the fourth order dissipation tensor defined as

$$\epsilon_{\alpha\beta}^{\gamma\delta} \equiv 2\nu \partial_\delta v'_\alpha \partial_\gamma v'_\beta \quad (9)$$

The properties of the source/sink terms on the right hand side need to be established before closure expressions can be constructed.

### 1.2.1 Equivalent forms of the pressure correlations.

The pressure correlations  $S_{\alpha\beta}^4$  can be given in several equivalent forms. It is instructive to split them analogous to the pressure-rate of strain correlations in the stress equations. It follows from (6) that

$$S_{\alpha\beta}^4 = -\frac{2\nu}{\rho} \{ \partial_\alpha \langle \partial_\gamma p' \partial_\gamma v'_\beta \rangle + \partial_\beta \langle \partial_\gamma p' \partial_\gamma v'_\alpha \rangle \} + B_{\alpha\beta}$$

where the non-gradient part is defined by

$$B_{\alpha\beta} \equiv \frac{2\nu}{\rho} \langle \partial_\gamma p' \partial_\gamma (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle$$

The non-gradient part  $B_{\alpha\beta}$  resembles the pressure-rate of strain correlation and shares with it the property

$$B_{\alpha\alpha} = 0$$

It follows that  $B_{\alpha\beta}$  redistributes intensity among the components of the dissipation tensor and leaves its trace unaffected. Further splitting of  $B_{\alpha\beta}$  leads to

$$B_{\alpha\beta} = \frac{2\nu}{\rho} \partial_\gamma \langle \partial_\gamma p' (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle - \frac{2\nu}{\rho} \langle \partial_{\gamma\gamma}^2 p' (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle$$

The non-diffusive part of  $B_{\alpha\beta}$  contains the Laplacian of the pressure fluctuations which is governed by

$$-\frac{1}{\rho} \partial_{\gamma\gamma}^2 p' = \partial_\beta v'_\alpha \partial_\alpha v'_\beta + 2\partial_\alpha v'_\beta \partial_\beta \langle v_\alpha \rangle - \partial_{\alpha\beta}^2 \langle v'_\alpha v'_\beta \rangle$$

This equation has an important consequence: The non-diffusive part of  $B_{\alpha\beta}$  can be represented locally in terms of velocity fluctuations. We get

$$B_{\alpha\beta} = \frac{2\nu}{\rho} \partial_\gamma \langle \partial_\gamma p' (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle +$$

$$2\nu \langle \partial_\gamma v'_\delta \partial_\delta v'_\gamma (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle + 4\nu \langle \partial_\gamma v'_\delta (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha) \rangle \partial_\delta \langle v_\gamma \rangle$$

We conclude that the non-diffusive and non-gradient part of the pressure correlations does not contain a direct influence of the wall and the wall effects can be represented as gradients and divergence of a flux. This property very important for the modelling effort. Inspection of the local part of  $B_{\alpha\beta}$  shows that it has the structure of the primary production term (5). Recasting this part in terms of vorticity and strain rate defined by

$$\omega_\alpha \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma} \partial_\beta v'_\gamma$$

and

$$s_{\alpha\beta} \equiv \frac{1}{2} (\partial_\alpha v'_\beta + \partial_\beta v'_\alpha)$$

leads to

$$2\nu\langle\partial_\gamma v'_\delta\partial_\delta v'_\gamma(\partial_\alpha v'_\beta + \partial_\beta v'_\alpha)\rangle = 4\nu\langle s'_{\alpha\beta}s'_{\gamma\delta}s'_{\gamma\delta}\rangle - 8\nu\langle s'_{\alpha\beta}\omega'_\gamma\omega'_\gamma\rangle$$

whereas the primary production terms appear as

$$S_{\alpha\beta}^3 = -4\nu(\langle s'_{\alpha\gamma}s'_{\gamma\delta}s'_{\delta\beta}\rangle + \epsilon_{\gamma\alpha\eta}\langle s'_{\beta\delta}s'_{\delta\gamma}\omega'_\eta\rangle + \epsilon_{\gamma\beta\eta}\langle s'_{\alpha\delta}s'_{\delta\gamma}\omega'_\eta\rangle + \epsilon_{\gamma\alpha\omega}\epsilon_{\delta\beta\eta}\langle s'_{\gamma\delta}\omega'_\eta\omega'_\omega\rangle)$$

The last term can be recast in terms of the Kronecker delta using the tensor relation

$$\epsilon_{\gamma\alpha\omega}\epsilon_{\delta\beta\eta} = \det \begin{pmatrix} \delta_{\gamma\delta} & \delta_{\gamma\beta} & \delta_{\gamma\eta} \\ \delta_{\alpha\delta} & \delta_{\alpha\beta} & \delta_{\alpha\eta} \\ \delta_{\omega\delta} & \delta_{\omega\beta} & \delta_{\omega\eta} \end{pmatrix}$$

leading to

$$S_{\alpha\beta}^3 = -4\nu(\langle s'_{\alpha\gamma}s'_{\gamma\delta}s'_{\delta\beta}\rangle + \epsilon_{\gamma\alpha\eta}\langle s'_{\beta\delta}s'_{\delta\gamma}\omega'_\eta\rangle + \epsilon_{\gamma\beta\eta}\langle s'_{\alpha\delta}s'_{\delta\gamma}\omega'_\eta\rangle + \langle s'_{\gamma\beta}\omega'_\alpha\omega'_\gamma\rangle + \langle s'_{\gamma\alpha}\omega'_\beta\omega'_\gamma\rangle - \langle s'_{\alpha\beta}\omega'_\gamma\omega'_\gamma\rangle - \delta_{\alpha\beta}\langle s'_{\gamma\eta}\omega'_\gamma\omega'_\eta\rangle)$$

It is apparent that no complete cancellation of triple correlations takes place. The pressure correlations appear now as a combination of the divergence of a flux and sources.

$$S_{\alpha\beta}^4 = \partial_\delta F_{\alpha\beta\delta}^p + Q_{\alpha\beta}^p$$

where the flux is defined by

$$F_{\alpha\beta\delta}^p \equiv \frac{2\nu}{\rho}(\epsilon_{\omega\delta\eta}\epsilon_{\omega\gamma\alpha}\langle\partial_\gamma p'\partial_\eta v'_\beta\rangle + \epsilon_{\omega\delta\eta}\epsilon_{\omega\gamma\beta}\langle\partial_\gamma p'\partial_\eta v'_\alpha\rangle)$$

or

$$F_{\alpha\beta\delta}^p = \frac{2\nu}{\rho}(\langle\partial_\delta p'(\partial_\alpha v'_\beta + \partial_\beta v'_\alpha)\rangle - \delta_{\alpha\delta}\langle\partial_\gamma p'\partial_\gamma v'_\beta\rangle - \delta_{\beta\delta}\langle\partial_\gamma p'\partial_\gamma v'_\alpha\rangle)$$

The trace of the flux  $F_{\alpha\beta\delta}^p$  is not zero but given by

$$F_{\alpha\alpha\delta}^p = -\frac{4\nu}{\rho}\langle\partial_\gamma p'\partial_\gamma v'_\delta\rangle$$

The redistributive sources are given by

$$Q_{\alpha\beta}^p \equiv 2\nu\langle\partial_\gamma v'_\delta\partial_\delta v'_\gamma(\partial_\alpha v'_\beta + \partial_\beta v'_\alpha)\rangle + 4\nu\langle\partial_\gamma v'_\delta(\partial_\alpha v'_\beta + \partial_\beta v'_\alpha)\rangle\partial_\delta\langle v_\gamma\rangle$$

where the trace of  $Q_{\alpha\beta}^p$  is zero. It is noteworthy that one of the components of  $\langle\partial_\gamma p'\partial_\eta v'_\alpha\rangle$  can be expressed in terms of a component of the dissipation tensor

$$4\frac{\nu}{\rho}\langle\partial_2 p'\partial_2 v'_1\rangle = \nu\partial_2\langle\epsilon_{12}\rangle + O(y)$$

according to the power series for the pressure given below (19). For  $\gamma \neq 2$  the gradient of the wall pressure can be expressed in terms of velocity using the momentum balances. In fact

$$\frac{1}{\rho} \partial_\gamma p' = \nu \partial_{yy}^2 v'_\gamma$$

holds at the wall  $y = 0.0$ . It follows that the wall pressure does not exert a direct influence on the dissipation rates if the expansion is carried out to second order. It is clear that modelling can be based on the properties of the flux and the redistributive source terms. However, the growth rates for the terms in the different formulations of the pressure correlations decide in the end their usefulness. This will be investigated in the following chapter.

### 1.2.2 Taylor series expansions for the near wall region.

The near wall region can be analyzed with Taylor series. The coordinate system is assumed to be located at the wall and  $x_2$  is the direction of the wall normal pointing into the flow field. It is convenient to rename the coordinates and variables as follows:  $x_1 \equiv x$ ,  $x_2 \equiv y$ ,  $x_3 \equiv z$  and  $v_1 \equiv u$ ,  $v_2 \equiv v$ ,  $v_3 \equiv w$ . The velocity components can be expanded with respect to the wall normal  $y$

$$u(x, y, z, t) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + O(y^4) \quad (10)$$

$$v(x, y, z, t) = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + O(y^4) \quad (11)$$

$$w(x, y, z, t) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + O(y^4) \quad (12)$$

where the coefficients are stochastic functions of  $x, z, t$  but not  $y$ . They are defined by

$$a_j(x, z, t) \equiv \frac{1}{j!} \frac{\partial^j u}{\partial y^j}(0)$$

$$b_j(x, z, t) \equiv \frac{1}{j!} \frac{\partial^j v}{\partial y^j}(0)$$

$$c_j(x, z, t) \equiv \frac{1}{j!} \frac{\partial^j w}{\partial y^j}(0)$$

The noslip condition at the wall implies that

$$a_0 = b_0 = c_0 = 0 \quad (13)$$

holds and mass balance

$$\partial_\alpha v_\alpha = 0 \quad (14)$$

leads to

$$\partial_y v_0 = b_1 = 0 \quad (15)$$

and

$$\partial_x a_n + \partial_z c_n = -(n+1)b_{n+1}$$

for  $n = 1, 2, \dots$ . The Taylor series for a fixed wall without suction or blowing are therefore given by

$$u(x, y, z, t) = a_1 y + a_2 y^2 + a_3 y^3 + O(y^4) \quad (16)$$

$$v(x, y, z, t) = b_2 y^2 + b_3 y^3 + O(y^4) \quad (17)$$

$$w(x, y, z, t) = c_1 y + c_2 y^2 + c_3 y^3 + O(y^4) \quad (18)$$

The expansion for the pressure can be given as follows

$$p(x, y, z, t) = p_0 + 2\mu b_2 y + 3\mu b_3 y^2 + p_3 y^3 + O(y^4) \quad (19)$$

where  $\mu$  denotes the dynamic viscosity and

$$p_j(x, z, t) \equiv \frac{1}{j!} \frac{\partial^j p}{\partial y^j}(0)$$

and the momentum balance normal to the wall was applied to a point at the wall. The expansion for the pressure gradient can be shown to be

$$\partial_\gamma p = 2\mu \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + 2\mu y \begin{pmatrix} \partial_1 b_2 \\ 3b_3 \\ \partial_3 b_2 \end{pmatrix} + 3y^2 \begin{pmatrix} \mu \partial_1 b_3 \\ p_3 \\ \mu \partial_3 b_3 \end{pmatrix} + O(y^3)$$

which shows that the terms up to first order are proportional to viscosity. The components of the Reynolds stress tensor appear in expanded form as

$$\langle u^2 \rangle = y^2 \langle a_1^2 \rangle + 2y^3 \langle a_1 a_2 \rangle + y^4 (2\langle a_1 a_3 \rangle + \langle a_2^2 \rangle) + O(y^5) \quad (20)$$

$$\langle v^2 \rangle = y^4 \langle b_2^2 \rangle + 2y^5 \langle b_2 b_3 \rangle + O(y^6) \quad (21)$$

$$\langle w^2 \rangle = y^2 \langle c_1^2 \rangle + 2y^3 \langle c_1 c_2 \rangle + y^4 (2\langle c_1 c_3 \rangle + \langle c_2^2 \rangle) + O(y^5) \quad (22)$$

$$\langle uv \rangle = y^3 \langle a_1 b_2 \rangle + y^4 (\langle a_1 b_3 \rangle + \langle a_2 b_2 \rangle) + O(y^5) \quad (23)$$

The components of the dissipation rate tensor vary near the wall according to

$$\begin{aligned} \langle \epsilon_{11} \rangle &= 2\nu \{ \langle a_1^2 \rangle + 4y \langle a_1 a_2 \rangle \\ &+ y^2 (6\langle a_1 a_3 \rangle + 4\langle a_2^2 \rangle + \langle (\partial_x a_1)^2 \rangle + \langle (\partial_z a_1)^2 \rangle) + O(y^3) \} \end{aligned} \quad (24)$$

$$\langle \epsilon_{22} \rangle = 2\nu \{ 4y^2 \langle b_2^2 \rangle + 12y^3 \langle b_2 b_3 \rangle + O(y^4) \} \quad (25)$$

$$\langle \epsilon_{33} \rangle = 2\nu \{ \langle c_1^2 \rangle + 4y \langle c_1 c_2 \rangle$$



$$+y^2(6\langle c_1 c_3 \rangle + 4\langle c_2^2 \rangle + \langle (\partial_x c_1)^2 \rangle + \langle \partial_x c_1 \rangle) + O(y^3)\} \quad (26)$$

$$\langle \epsilon_{12} \rangle = 2\nu\{2y\langle a_1 b_2 \rangle + y^2(4\langle a_2 b_2 \rangle + 3\langle a_1 b_3 \rangle) + O(y^3)\} \quad (27)$$

It is apparent from these series that different components grow differently near the wall. It follows that the boundary values for the dissipation rate tensor are given by

$$\langle \epsilon_{11} \rangle(0) = 2\nu\langle a_1^2 \rangle \quad (28)$$

$$\langle \epsilon_{22} \rangle(0) = 0.0 \quad (29)$$

$$\langle \epsilon_{33} \rangle(0) = 2\nu\langle c_1^2 \rangle \quad (30)$$

$$\langle \epsilon_{12} \rangle(0) = 0.0 \quad (31)$$

It can be shown that the anisotropy of the Reynolds stress tensor is equal to the anisotropy of the dissipation rate tensor at the wall and that the normal derivative at the wall of the anisotropy tensor of the dissipation rate is twice the normal derivative for the stress tensor. We consider now the near wall variation of the individual terms in the dissipation rate equations.

### 1.2.2.1 Viscous Diffusion.

The dominant term in viscous diffusion is the normal derivative given by

$$\partial_y(\nu\partial_y\langle\epsilon_{11}\rangle) = 4\nu^2(6\langle a_1 a_3 \rangle + 4\langle a_2^2 \rangle + \langle (\partial_x a_1)^2 \rangle + \langle (\partial_x a_1) \rangle) + O(y) \quad (32)$$

$$\partial_y(\nu\partial_y\langle\epsilon_{22}\rangle) = 16\nu^2\langle b_2^2 \rangle + O(y) \quad (33)$$

$$\partial_y(\nu\partial_y\langle\epsilon_{33}\rangle) = 4\nu^2(6\langle c_1 c_3 \rangle + 4\langle c_2^2 \rangle + \langle (\partial_x c_1)^2 \rangle + \langle (\partial_x c_1) \rangle) + O(y) \quad (34)$$

$$\partial_y(\nu\partial_y\langle\epsilon_{12}\rangle) = 4\nu^2(4\langle a_2 b_2 \rangle + 3\langle a_1 b_3 \rangle) + O(y) \quad (35)$$

and they emerge as terms of order unity near the wall for all components.

### 1.2.2.2 Turbulent diffusion.

Turbulent diffusion of the dissipation rate component  $\epsilon_{\alpha\beta}$  is defined by

$$\partial_\gamma F_\gamma^{\alpha\beta} \equiv -\partial_\gamma \langle v'_\gamma \epsilon_{\alpha\beta} \rangle \quad (36)$$

For the case of boundary layer type flows only the flux normal to the wall is relevant. The series expansions lead then to the following expressions for the components of the dissipation rate tensor

$$\partial_y F_y^{11} = 2y \langle b_2 \epsilon_{11}(0) \rangle + O(y^2) \quad (37)$$

$$\partial_y F_y^{22} = 4y^3 \langle b_2 \epsilon_{22}(0) \rangle + O(y^4) \quad (38)$$

$$\partial_y F_y^{33} = 2y \langle b_2 \epsilon_{33}(0) \rangle + O(y^2) \quad (39)$$

$$\partial_y F_y^{12} = 3y^2 \langle b_2 \partial_y \epsilon_{12}(0) \rangle + O(y^3) \quad (40)$$

The expansions show that turbulent diffusion is not of leading order near the wall.

### 1.2.2.3 Secondary production $S_{\alpha\beta}^1$ .

The interaction of the mean rate of strain and the dissipation rate tensor acts as production for the dissipation rates in the same fashion as the Reynolds stress and mean strain rates for the stresses. There is however a fundamental difference between this type of production for dissipation rates and stresses: It is of leading order for the stress balance but of second order for the dissipation rates for high Reynolds number flows. The situation near the wall is entirely different. The components of  $S_{\alpha\beta}^1$  turn out to grow with wall distance as follows

$$S_{11}^1 = -2y \partial_y \langle u \rangle(0) \partial_y \langle \epsilon_{12} \rangle(0) + O(y^2) \quad (41)$$

$$S_{22}^1 = -16\nu y^3 \partial_{yy}^2 \langle v \rangle(0) \langle b_2^2 \rangle + O(y^4) \quad (42)$$

$$S_{33}^1 = 0.0 \quad (43)$$

$$S_{12}^1 = -8\nu y^2 \partial_y \langle u \rangle(0) \langle b_2^2 \rangle + O(y^3) \quad (44)$$

The secondary production terms  $S_{\alpha\beta}^1$  are not of leading order near the wall, but closed. Hence, they need not be neglected.

### 1.2.2.4 Secondary production $S_{\alpha\beta}^2$ .

The series expansions lead to the following results

$$S_{11}^2 = -y \{ \partial_y \langle u \rangle(0) \partial_x \langle \epsilon_{11} \rangle(0) + 2\partial_{yy}^2 \langle v \rangle(0) \langle \epsilon_{11} \rangle(0) \} + O(y^2) \quad (45)$$

$$S_{22}^2 = -2\nu y^3 \{ 2\partial_y \langle u \rangle(0) \partial_x \langle b_2^2 \rangle + 8\partial_{yy}^2 \langle v \rangle(0) \langle b_2^2 \rangle \} + O(y^4) \quad (46)$$

$$S_{33}^2 = -y \{ \partial_y \langle u \rangle(0) \partial_x \langle \epsilon_{33} \rangle(0) + 2\partial_{yy}^2 \langle v \rangle(0) \langle \epsilon_{33} \rangle(0) \} + O(y^2) \quad (47)$$

$$S_{12}^2 = -2\nu y^2 \{2\partial_y \langle u \rangle(0)(\langle b_2 \partial_x a_1 \rangle + \langle a_1 \partial_x b_2 \rangle) + 4\partial_{yy}^2 \langle v \rangle(0)\langle a_1 b_2 \rangle\} + O(y^3) \quad (48)$$

It is clear from these expansions that the source terms  $S_{\alpha\beta}^2$  are of second or higher order for boundary layer flows.

#### 1.2.2.5 Primary production.

The primary production or vortex stretching terms are the dominant production terms in high Re-number flows. For the near wall region they appear in expanded form as

$$S_{11}^3 = -4\nu \{y(\frac{1}{2}\partial_x \langle a_1^3 \rangle + 3\langle a_1^2 b_2 \rangle) + O(y^2)\} \quad (49)$$

$$S_{22}^3 = -4\nu \{y^3(10\langle b_2^3 \rangle + \partial_x \langle a_1 b_2^2 \rangle) + O(y^4)\} \quad (50)$$

$$S_{33}^3 = -4\nu \{y(\frac{1}{2}\partial_x \langle a_1 c_1^2 \rangle + 3\langle c_1^2 b_2 \rangle) + O(y^2)\} \quad (51)$$

$$S_{12}^3 = -2\nu \{y^2(\partial_x \langle b_2 a_1^2 \rangle + 8\langle a_1 b_2^2 \rangle + 2\langle c_1 b_2 \partial_z a_1 \rangle + \langle a_1 c_1 \partial_z b_2 \rangle) + O(y^3)\} \quad (52)$$

The primary production is not of leading order near the wall, but grows with the same order as the secondary production term  $S_{\alpha\beta}^1$  with wall distance.

#### 1.2.2.6 Viscous destruction.

Viscous effects can destroy the rate of dissipation and this process is contained in  $D_{\alpha\beta}$ . The series analysis shows that the components of  $D_{\alpha\beta}$  are near the wall given by

$$D_{11} = 4\nu^2 \{4\langle a_2^2 \rangle + 2\langle (\partial_x a_1)^2 \rangle + 2\langle (\partial_z a_1)^2 \rangle\} + O(y) \quad (53)$$

$$D_{22} = 4\nu^2 \{4\langle b_2^2 \rangle + 24y\langle b_2 b_3 \rangle\} + O(y^2) \quad (54)$$

$$D_{33} = 4\nu^2 \{4\langle c_2^2 \rangle + 2\langle (\partial_x c_1)^2 \rangle + 2\langle (\partial_z c_1)^2 \rangle\} + O(y) \quad (55)$$

$$D_{12} = 4\nu^2 \{4\langle a_2 b_2 \rangle + y(4\langle \partial_x a_1 \partial_x b_2 \rangle + 4\langle \partial_z a_1 \partial_z b_2 \rangle + 12\langle a_2 b_3 \rangle + 12\langle a_3 b_2 \rangle)\} + O(y^2) \quad (56)$$

All components of  $D_{\alpha\beta}$  turn out to be of leading order.

#### 1.2.2.7 Pressure correlations.

The Taylor series for the pressure (19) contains viscous terms which are due to the momentum balances. Differentiation of (19) leads to expansions which contain viscous contributions in lowest order. The components of the pressure correlations  $S_{\alpha\beta}^4$  appear as follows

$$S_{11}^4 = 4\nu^2 \{6\langle a_1 b_3 \rangle + y(2\langle \partial_x a_1 \partial_x a_2 \rangle + 2\langle \partial_z a_1 \partial_z a_2 \rangle + 4\langle a_2 \partial_x b_2 \rangle + 6\langle a_1 \partial_x b_3 \rangle) + O(y^2)\} \quad (57)$$

$$S_{22}^4 = 4\nu^2 \{12y\langle b_2 b_3 \rangle + O(y^2)\} \quad (58)$$

$$S_{33}^4 = 4\nu^2 \{6\langle c_1 b_3 \rangle + y(2\langle \partial_x c_1 \partial_x a_2 \rangle + 2\langle \partial_z c_1 \partial_z b_2 \rangle + 4\langle c_2 \partial_x c_2 \rangle + 6\langle c_1 \partial_x b_3 \rangle) + O(y^2)\} \quad (59)$$

$$S_{12}^4 = 2\nu^2 \{6\langle a_1 b_3 \rangle + y(2\langle \partial_x a_1 \partial_x b_2 \rangle + 2\langle \partial_z a_1 \partial_z b_2 \rangle + 12\langle a_2 b_3 \rangle + 4\langle b_2 \partial_x b_2 \rangle + \frac{6}{\rho\nu} \langle a_1 p_3 \rangle) + O(y^2)\} \quad (60)$$

The series expansions of the pressure correlations lead to several important conclusion: The effect of the pressure correlations in lowest order is local in terms of velocity correlations. No Poisson integral appears in lowest order since the wall pressure does not appear in the lowest order terms. The split of the pressure correlations obtained in chapter 2.1 leads to a flux  $F_{\alpha\beta\gamma}^p$  such that the corresponding source term is strictly redistributive and local in terms of velocity fluctuations. The pressure flux is local in velocity in expanded form up to second order. Note that there is a clear advantage for not splitting the pressure correlations in the Reynolds stress equations near the wall since the split terms (rate of strain correlation and pressure transport) grow with different rates in the viscous sublayer. This is not the case for the pressure correlations in the dissipation rate equations. The components of the pressure flux emerge for the case of a flat plate boundary layer as

$$F_{112}^p = \frac{4\nu}{\rho} \langle \partial_2 p' \partial_1 v'_1 \rangle$$

$$F_{222}^p = \frac{4\nu}{\rho} \{ \langle \partial_3 p' \partial_3 v'_2 \rangle + \langle \partial_1 p' \partial_1 v'_2 \rangle \}$$

$$F_{332}^p = \frac{4\nu}{\rho} \langle \partial_2 p' \partial_3 v'_3 \rangle$$

$$F_{122}^p = \frac{2\nu}{\rho} \{ \langle \partial_2 p' \partial_1 v'_2 \rangle - \langle \partial_3 p' \partial_3 v'_1 \rangle \}$$

which can be analyzed with the aid of Taylor series. We get the following estimates for the divergence of this flux near the wall

$$\partial_2 F_{112}^p = 8\nu^2 \langle b_2 \partial_x a_1 \rangle + O(y)$$

$$\begin{aligned}
\partial_2 F_{222}^p &= O(y) \\
\partial_2 F_{332}^p &= 8\nu^2 \langle b_2 \partial_z c_1 \rangle + O(y) \\
\partial_2 F_{122}^p &= -4\nu^2 \langle c_2 \partial_z c_1 \rangle + O(y)
\end{aligned}$$

It follows that

$$\partial_2 F_{112}^p + \partial_2 F_{332}^p = -\partial_2(\nu \partial_2 \langle \epsilon_{22} \rangle) + O(y)$$

holds. This proves that the pressure flux terms are of leading order near the wall because viscous diffusion is of leading order in this region.

### 1.2.2.8 Near wall production.

The production terms proportional to the curvature of the mean velocity profile are responsible for additional production in the near wall region. The series expansions lead to

$$S_{11}^5 = 4\nu \{ y \langle a_1^2 \rangle \partial_{xy}^2 \langle u \rangle(0) + y^2 \langle a_1^2 \rangle \partial_{xyy}^3 \langle u \rangle(0) + 3 \langle a_1 a_2 \rangle \partial_{xy}^2 \langle u \rangle(0) + 2 \langle a_2 b_2 \rangle \partial_{yy}^2 \langle u \rangle(0) \} + O(y^3) \quad (61)$$

$$S_{22}^5 = O(y^3) \quad (62)$$

$$S_{33}^5 = 0 \quad (63)$$

$$S_{12}^5 = \nu y^2 \{ \langle a_1^2 \rangle \partial_{xyy}^3 \langle u \rangle(0) + \langle a_1 b_2 \rangle \partial_{xyy}^3 \langle u \rangle(0) \} + O(y^3) \quad (64)$$

The near wall production emerges as second order effect near the wall if the boundary layer assumptions are satisfied. The component  $\langle \epsilon_{11} \rangle$  receives all the energy in lowest order.

### 1.2.2.9 Transport equations in lowest order.

The series expansions for the source and diffusive terms allow the set up of the transport equations for the components of the dissipation rate tensor in lowest order. It turns out that all equations are of the same zeroth order.

$$\partial_t \langle \epsilon_{11} \rangle = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{11} \rangle) - D_{11} + S_{11}^4 \quad (65)$$

$$0 = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{22} \rangle) - D_{22} \quad (66)$$

$$\partial_t \langle \epsilon_{33} \rangle = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{33} \rangle) - D_{33} + S_{33}^4 \quad (67)$$

$$0 = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{12} \rangle) - D_{12} + S_{12}^4 \quad (68)$$

These equations are valid near the wall provided none of the surviving correlations vanishes. For steady flows the following equations hold then at the wall

$$D_{11} - S_{11}^4 = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{11} \rangle)$$

$$D_{22} = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{22} \rangle)$$

$$D_{33} - S_{33}^4 = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{33} \rangle)$$

$$D_{12} - S_{12}^4 = \partial_\gamma (\nu \partial_\gamma \langle \epsilon_{12} \rangle)$$

which can be combined with the limit relations for the stress balances to establish constraints for the modelled terms at the wall.

### 1.3 Closure model for the dissipation rate equations.

The series expansions for the near wall properties of the dissipation rate equations can be used to analyze and to modify closure expressions. In several cases no model expressions exist and new models will be developed and analyzed.

#### 1.3.1 Time scales.

Several time scales can be constructed for the near wall region with the aid of the dissipation rate tensor. First we note that a scale dependent on the wall normal vector can be obtained in the form

$$\tau \equiv \frac{k}{n_\gamma n_\delta \langle \epsilon_{\gamma\delta} \rangle} \quad (69)$$

If  $n_\gamma = \delta_{\gamma 2}$  it follows that this time scale is given by

$$\tau = \frac{k}{\langle \epsilon_{22} \rangle}$$

and the series expansions show that both numerator and denominator depend quadratically on wall distance. The wall limit is in fact a nonzero value given by

$$\tau(0) = \frac{\langle \partial_y u_0 \partial_y u_0 \rangle + \langle \partial_y w_0 \partial_y w_0 \rangle}{4\nu \langle \partial_{yy}^2 v_0 \partial_{yy}^2 v_0 \rangle} \quad (70)$$

Hence, a time scale with a nonzero limit at the fixed wall was constructed. This time scale avoids the problem associated with the modified (also called homogeneous) dissipation rate

$$\tilde{\epsilon} \equiv \epsilon - 2\nu(\partial_y \sqrt{k})^2$$

which may change sign in the flow field. The inverse of another time scale with tensorial character using the dissipation rate tensor can be set up as follows

$$\tau_{\alpha\beta}^{-1} \equiv \frac{1}{\epsilon} \langle \epsilon_{\alpha\gamma} \rangle \langle v'_\gamma v'_\delta \rangle^{-1} \langle \epsilon_{\delta\beta} \rangle \quad (71)$$

where  $\langle v'_\gamma v'_\delta \rangle^{-1}$  denotes the inverse of the Reynolds stress tensor. Conversely is a time scale given by

$$\tau_{\alpha\beta} \equiv \frac{1}{k} \langle v'_\alpha v'_\gamma \rangle \langle \epsilon_{\gamma\delta} \rangle^{-1} \langle v'_\delta v'_\beta \rangle \quad (72)$$

where  $\langle \epsilon_{\alpha\beta} \rangle^{-1}$  denotes now the inverse of the dissipation rate tensor.

#### 1.3.2 Turbulent diffusion.

The gradient flux model for  $F_\gamma^{\alpha\beta}$  can be given in the form

$$F_\gamma^{\alpha\beta} \doteq c_s \frac{k}{\epsilon} \langle v'_\gamma v'_\delta \rangle \partial_\delta \langle \epsilon_{\alpha\beta} \rangle \quad (73)$$

where the kinetic energy is denoted by

$$k(\underline{x}, t) \equiv \frac{1}{2} \langle v'_\alpha v'_\alpha \rangle$$

and the dissipation rate  $\epsilon$  by

$$\epsilon(\underline{x}, t) \equiv \frac{1}{2} \langle \epsilon_{\alpha\alpha} \rangle$$

The constant  $c_s$  has values in the range 0.15 – 0.18. The near wall properties of this model follow from the series expansions (10) to (27) as

$$\partial_y F_y^{11} = O(y^5)$$

whereas the exact term has a first order variation with respect to the wall distance according to (37). Similar discrepancies are observed for the other components. It is clear that model expressions developed for the high Re-number regime will not represent the near wall region properly. The present model (73) implies that near the wall turbulent diffusion is essentially neglected in comparison to the exact term. Inspection of the model (73) shows that there are two reasons for its failure near the wall: The time scale  $\frac{k}{\epsilon}$  approaches zero at the wall and the diffusivity is solely determined by the normal stress component  $\langle (v')^2 \rangle$  which varies as  $y^4$  near the wall. The situation can be improved if a composite time scale that approaches the scale defined by (69) near the wall is used. The modified closure model is then given by

$$F_\gamma^{\alpha\beta} \doteq \frac{2}{3} c_s \frac{k}{n_\eta n_\omega \langle \epsilon_{\eta\omega} \rangle} \langle v'_\gamma v'_\delta \rangle \partial_\delta \langle \epsilon_{\alpha\beta} \rangle \quad (74)$$

The factor 2/3 results from the requirement that the high Re- number limit of the time scale must agree with  $k/\epsilon$ .

A different model that satisfies all growth estimates and has the correct tensorial and dimensional properties can be constructed if the dependence of the turbulent flux on the wall parameters  $\underline{n}$  and  $R_{et}$  is taken into account and the near wall model is combined with the high  $R_e$ -number model. The near wall model is given by

$$-\langle v'_\gamma \epsilon_{\alpha\beta} \rangle \doteq c'_e f_{w2}(R_{et}) [n_\eta n_\zeta \langle v'_\eta v'_\zeta \rangle]^{\frac{1}{2}} n_\gamma \langle \epsilon_{\alpha\beta} \rangle$$

It is straightforward to check that it has the same growth rate and the same tensorial properties as the exact term. The model represents turbulent transport towards the wall and has the form of a convective term. There exist several high Reynolds number models for the

turbulent flux of dissipation rate (Hanjalic and Launder, 1972, Lumley, 1978). The present model is an analogue of the flux model for the stresses. It is given by

$$-\langle v'_\gamma \epsilon_{\alpha\beta} \rangle \doteq c_e \frac{k}{\epsilon} \langle v'_\gamma v'_\delta \rangle \frac{\partial}{\partial x_\delta} \langle \epsilon_{\alpha\beta} \rangle$$

The combined closure model is the set up as follows

$$-\langle v'_\gamma \epsilon_{\alpha\beta} \rangle \doteq c'_e f_{w2}(R_{et}) [n_\eta n_\zeta \langle v'_\eta v'_\zeta \rangle]^{\frac{1}{2}} n_\gamma \langle \epsilon_{\alpha\beta} \rangle + c_e (1 - f_{w2}(R_{et})) \frac{k}{\epsilon} \langle v'_\gamma v'_\delta \rangle \frac{\partial}{\partial x_\delta} \langle \epsilon_{\alpha\beta} \rangle$$

where  $f_{w2}(R_{et})$  denotes a function of the local Reynolds number such that  $f_{w2}$  goes to unity as the wall is approached and to zero in the turbulent zone. Furthermore, The function  $f_{w2}(R_{et})$  should be nonnegative and it should not modify the dependence on the wall distance for the near wall model. It follows that  $f_{w2}$  must be an exponential function of the Reynolds number given by

$$f_{w2}(R_{et}) = \exp[-(\frac{R_{et}}{R_{e2}})^2]$$

where  $R_{e2}$  is a constant measuring the range of influence for the near wall model. This function of the Reynolds number has the well known property that all its derivatives at zero vanish. Hence, it does not modify the growth rate of the near wall model.

### 1.3.3 Secondary production $S^2_{\alpha\beta}$ .

The properties of  $S^2_{\alpha\beta}$  in the high and low Re-number limits will be considered first. Local isotropy requires that

$$\lim_{Re \rightarrow \infty} \langle \epsilon^{\gamma\delta}_{\alpha\beta} \rangle = \frac{2}{9} \delta_{\alpha\beta} \delta_{\gamma\delta} \epsilon$$

holds. It follows that the secondary production terms  $S^2_{\alpha\beta}$  are for high Re-numbers given by

$$\lim_{Re \rightarrow \infty} S^2_{\alpha\beta} = -\frac{4}{9} \delta_{\alpha\beta} \epsilon \partial_\gamma \langle v_\gamma \rangle \quad (75)$$

The divergence of the mean velocity is zero for incompressible flows and it follows that the high Re-number form of the secondary production terms can be neglected. The near wall variation of the secondary production term  $S^2_{11}$  according to (45) can be regarded as

$$S^2_{11} = -\langle u \rangle \partial_x \langle \epsilon_{11} \rangle - 2\langle \epsilon_{11} \rangle \partial_y \langle v \rangle + O(y^2)$$

and it follows that this term is at best of the same order of magnitude as the mean convection term. Similar relations hold for the other components and we conclude that the secondary production terms  $S^2_{\alpha\beta}$  can be neglected.

### 1.3.4 Primary production and viscous destruction.



The closure model for these terms is fundamentally different in the high and the low Reynolds number limits. Both limits need to be considered and the corresponding closure models must be merged to cover the range of Reynolds numbers from zero to infinity.

#### High Reynolds number limit.

The order of magnitude estimates for primary production and viscous destruction at high Re-numbers shows that they are of leading order, but their difference is of the same order as the secondary production term  $S_{\alpha\beta}^1$ . It follows that they should not be treated separately but their difference should be modelled as function of the available information. The model consistent with second order moments is in general given by

$$S_{\alpha\beta}^3 - D_{\alpha\beta} \doteq \frac{\epsilon}{\tau} \Psi_{\alpha\beta}(\langle \epsilon_{\delta\omega} \rangle, \langle v'_\delta v'_\omega \rangle, \partial_\delta \langle v_\omega \rangle) \quad (76)$$

The dimensionless and symmetric tensor  $\Psi_{\alpha\beta}$  should represent both production due to the interaction of vorticity and strain rates and the destruction due to viscous effects. If we impose the condition of local isotropy on this model we get the following variant

$$\Psi_{\alpha\beta} \doteq \delta_{\alpha\beta} \Psi^{(1)}(\langle \epsilon_{\gamma\delta} \rangle, \langle v'_\gamma v'_\delta \rangle, \partial_\gamma \langle v_\delta \rangle) - \Psi_{\alpha\beta}^{(2)}(\langle \epsilon_{\delta\omega} \rangle, \langle v'_\delta v'_\omega \rangle, \partial_\delta \langle v_\omega \rangle)$$

where the first part represents the productive and the second the destructive contribution to the model for the difference of  $S_{\alpha\beta}^3$  and  $D_{\alpha\beta}$ . The model can be set up to be consistent with the standard expressions for the trace equation (see Launder, Reece and Rodi, 1976)

$$\Psi_{\alpha\beta} \doteq -c_{\epsilon 1} \frac{1}{3} \delta_{\alpha\beta} \frac{\langle v'_\gamma v'_\delta \rangle}{\epsilon} \partial_\gamma \langle v_\delta \rangle - c_{\epsilon 2} \frac{\langle \epsilon_{\alpha\beta} \rangle}{\epsilon}$$

and the time scale is given by

$$\tau = \frac{k}{\epsilon}$$

The closure model

$$S_{\alpha\beta}^3 - D_{\alpha\beta} \doteq -c_{\epsilon 1} \frac{1}{3} \delta_{\alpha\beta} \frac{\epsilon}{k} \langle v'_\gamma v'_\delta \rangle \partial_\gamma \langle v_\delta \rangle - c_{\epsilon 2} \langle \epsilon_{\alpha\beta} \rangle \frac{\epsilon}{k} \quad (77)$$

emerges. It is, however, not applicable to wall bounded flows since the destructive part of this model becomes singular as the wall is approached. This deficiency can be corrected either by defining a time scale that does not go to zero at the wall or by merging low and high Reynolds number closures with a Reynolds number dependent function such that the singularity is removed. Finally, we note that this model is not necessarily positive definite because the productive part  $\Psi^{(1)}$  is not positive definite, a property shared with the exact term. Refined closure models can be constructed using tensorial time scales introduced in chapter 1.3.1.

#### Low Reynolds number limit.

The dependence on the wall distance is the deciding property as the wall is approached. We recall that the primary production terms decay with wall distance, according to chapter 2.2.5, as follows

$$O(S_{11}^3) = O(S_{33}^3) = y$$

$$O(S_{22}^3) = y^3$$

$$O(S_{12}^3) = y^2$$

whereas the viscous destruction terms are all of leading order

$$O(D_{\alpha\beta}) = 1$$

according to chapter 2.2.6. It follows that they must be modelled separately in the near wall region. The first step in the construction of the low Reynolds number version is the analysis of the near wall properties of the high Re-number model. It follows from (77) that

$$O\left(\frac{\epsilon}{\tau}\Psi^{(1)}\right) = y$$

which is at variance with the detailed decay laws for  $S_{\alpha\beta}^3$  given above for  $\alpha = \beta = 2$  and  $\alpha = 1, \beta = 2$ . However, since the primary production is not of leading order near the wall it would be acceptable. A more serious problem arises in the destructive part  $\Psi_{\alpha\beta}^{(2)}$ . If we construct a time scale such as (69) for the near wall region we avoid the singularity, but it is not possible to satisfy the decay laws for the destructive terms  $D_{\alpha\beta}$ . It follows that the time scale in any closure of the form given by  $\Psi_{\alpha\beta}^{(2)}$  cannot be a scalar but has to be a tensor of rank two (or higher) with positive eigenvalues. It is not difficult to construct a closure model  $\Psi_{\alpha\beta}^{(2)}$  such that the decay law is satisfied. For instance, the model

$$D_{\alpha\beta} \approx -c_{\epsilon 2} \tau^6 \frac{\partial \epsilon_{\alpha\gamma}}{\partial x_\delta \partial x_\eta} \frac{\partial \epsilon_{\beta\gamma}}{\partial x_\delta \partial x_\eta}$$

with (69) as time scale possesses the correct tensor properties and the correct decay law as the wall is approached. However, it is unacceptable as closure model because it produces in regions where the second derivatives are all positive a second order pde with negative diffusivity. The initial/boundary value problem for such an equation is not well posed and the numerical solution futile. Furthermore, this model would not be realizable, because the limit  $\langle \epsilon_{\alpha\beta} \rangle \rightarrow 0$  does not imply that the second derivatives of  $\langle \epsilon_{\alpha\beta} \rangle$  go to zero. It follows that the closure model for the viscous destruction of the dissipation rate components must be of the form

$$D_{\alpha\beta} \approx F_{\alpha\beta}(\langle \epsilon_{\gamma\eta} \rangle, \langle v'_\gamma v'_\eta \rangle, \partial_\zeta \langle \epsilon_{\gamma\eta} \rangle, \partial_\zeta \langle v'_\gamma v'_\eta \rangle, Re_t)$$

where  $Re_t$  denotes the turbulent Reynolds number defined by

$$Re_t \equiv \frac{k^2}{\epsilon \nu} \tag{78}$$

The example cited above for the destruction of dissipation rate showed that derivatives lower than second should be used. Several closure models can be set up that are tensorially consistent, have the correct decay law near the wall and avoid the stability problems incurred by negative diffusivities. We note two of them, first

$$D_{\alpha\beta} \doteq -c_{\epsilon 2} \frac{\langle \epsilon_{\alpha\beta} \rangle}{\tau} - c_{\epsilon 3} \tau^2 n_\gamma n_\delta \frac{\partial \langle \epsilon_{\alpha\eta} \rangle}{\partial x_\gamma} \frac{\partial \langle \epsilon_{\beta\eta} \rangle}{\partial x_\delta} \quad (79)$$

and

$$D_{\alpha\beta} \doteq -c_{\epsilon 2} \frac{\langle \epsilon_{\alpha\beta} \rangle}{\tau} - c_{\epsilon 3} \nu \epsilon n_\gamma n_\delta \frac{\partial}{\partial x_\gamma} \left( \frac{\langle \epsilon_{\alpha\eta} \rangle}{\epsilon} \right) \frac{\partial}{\partial x_\delta} \left( \frac{\langle \epsilon_{\beta\eta} \rangle}{\epsilon} \right) \quad (80)$$

as second variant. Note that  $n_\alpha$  denotes the unit normal vector of the wall pointing into the flow field. The time scale  $\tau$  is given by (69) to avoid a singularity at the wall. The two models look very similar, but inspection of the time scale (69) shows at once that the first model is most likely unstable. Suppose that kinetic energy  $k$  and dissipation rate component  $n_\gamma n_\delta \langle \epsilon_{\gamma\delta} \rangle = \langle \epsilon_{22} \rangle$  are in equilibrium, then assume that this equilibrium is disturbed, say the kinetic energy is reduced by a small amount. It follows from the fact that the second part in the first model is quadratically proportional to the kinetic energy that the rate of destruction of the dissipation rate components is decreased by the disturbance and consequently are the normal stresses further decreased and the equilibrium state is not recovered. Hence is this model unstable. It follows that the second model is the preferred one.

#### Merger of low and high Re-number models.

The closure models for the destruction term (79) or (80) valid for the limit  $Re \rightarrow 0$  and the destructive part of (77) valid for the high Re number limit must be merged together to produce a model valid throughout the flow field. Suppose both limit expressions have decay laws near the wall that do not need change via the function weighting them according to the local Reynolds number (or any other function proportional to the distance to the wall). Then we need a weight function that does not change the dependence on wall distance near the wall. This implies that we must find a function which has zero derivatives at zero. It is well known that the exponential function

$$f(y) = \exp\left(-\frac{1}{y^2}\right)$$

has the desired property, in fact, it is infinitely often differentiable but not analytic at zero (its Taylor series is identically zero at  $y = 0$ ). Hence, we can establish a low Reynolds number function

$$f(Re) = \exp\left[-\left(\frac{Re^0}{Re}\right)^2\right]$$

which is zero at zero Re-number and unity at infinite Re-number. The constant  $Re^0$  determines the range of Reynolds numbers for which the function is close to zero. Other functions have been proposed that are proportional to some power of the wall distance and change therefore the decay law. Several models have been proposed for the functions  $f_1$ , multiplying

the production of dissipation rate, and  $f_2$ , multiplying the destruction of dissipation rate. Vandromme et al. (1983) suggested

$$f_1 = 1.0$$

and

$$f_2 = 1.0 - 0.22 \exp\left\{-\left(\frac{Re_t}{6}\right)^2\right\}$$

based on the original model of Hanjalic and Launder (1976) and obtained good agreement with measurements in flat plate boundary layers. Recent developments surveyed by Launder (1989) use the second and third invariants of the Reynolds stress tensor to represent the wall influence on production and destruction processes.

### 1.3.5 Pressure correlations.

It was shown in chapter 1.2 that the pressure correlations can be split into transport and source terms such that the source terms are strictly redistributive and have no effect on the trace of the dissipation rate tensor in analogy to the pressure-strain correlations for the Reynolds stress tensor. It follows that they must redistribute intensity among the components of the dissipation tensor. Kolmogorov's hypothesis of local isotropy requires that the dissipation rate tensor approaches its isotropic form as the Reynolds number approaches infinity. It follows that a return to isotropy model given by

$$Q_{\alpha\beta}^p \doteq -\frac{c_4}{\tau}(\langle\epsilon_{\alpha\beta}\rangle - \frac{2}{3}\delta_{\alpha\beta}\epsilon) \quad (79)$$

would satisfy this condition. The open question is the time scale. The scale

$$\tau = \frac{k}{\epsilon}$$

is the obvious choice for the high Re-number limit, but the model becomes incorrect as the wall is approached because  $\tau^{-1}$  goes to infinity with  $y^{-2}$ . Modification of the time scale according to (69) solves this problem and

$$Q_{\alpha\beta}^p \doteq -c_4 \frac{n_\gamma n_\delta \langle\epsilon_{\gamma\delta}\rangle}{k}(\langle\epsilon_{\alpha\beta}\rangle - \frac{2}{3}\delta_{\alpha\beta}\epsilon) \quad (80)$$

emerges as nonsingular variant. The transport part of the pressure correlation can be modelled in terms of the viscous diffusion terms because the wall limits indicate this form. The model

$$F_{\alpha\beta\gamma}^p \doteq -\frac{\nu}{4}\partial_\delta\langle\epsilon_{\gamma\delta}\rangle n_\delta\{\epsilon_{\alpha\eta\omega}t_\beta b_\omega + \epsilon_{\beta\eta\omega}t_\alpha b_\omega - \epsilon_{\alpha\eta\omega}t_\omega b_\beta - \epsilon_{\beta\eta\omega}t_\omega b_\alpha\} \quad (81)$$

where the tangential unit vector is defined by

$$t_\alpha \equiv \frac{\langle v_\alpha \rangle(\delta)}{|\langle v_\beta \rangle(\delta)|} \quad (82)$$

and the binormal by

$$b_\alpha \equiv \epsilon_{\alpha\beta\gamma} t_\beta n_\gamma$$

satisfies the near wall properties for the components  $F_{112}^p$  and  $F_{332}^p$  and neglects the components  $F_{222}^p$  and  $F_{122}^p$ . The effect of the diffusive part of the pressure correlations is therefore inhibition of the viscous diffusion near the wall for the diagonal components corresponding to motion parallel to the wall. The effect on shear component and the diagonal component corresponding to the motion normal to the wall are neglected.

### 1.3.6 Near wall production.

The near wall production requires in general flows a closure model. However, for the near wall region in boundary layers expressions can be given that are exact in the wall limit.

$$S_{11}^5 \doteq -4\nu\partial_2\langle v_1'v_2'\rangle\partial_{22}^2\langle v_1\rangle \quad (83)$$

$$S_{22}^5 = S_{33}^5 = 0.0 \quad (84)$$

$$S_{12}^5 \doteq -\nu\partial_2\langle v_2'^2\rangle\partial_{22}^2\langle v_1\rangle \quad (85)$$

The model assumption is essentially the assumed validity of these expressions for finite distance from the wall. Since all expressions are proportional to the laminar viscosity, quick decay with increasing wall distance can be expected.

## 2.0 Reynolds stress and complete second order models.

The usual closure models based on the trace of the dissipation rate tensor will be considered first. Several versions of the Reynolds stress model are available and the most reliable stress model (Vandromme et al., 1983) will be evaluated. It will serve as test bed for the complete second order closure.

### 2.1 Stress equations.

The balance for the Reynolds stress components is given by

$$\begin{aligned} \langle \rho \rangle \left( \frac{\partial}{\partial t} + \langle v_\gamma \rangle \frac{\partial}{\partial x_\gamma} \right) \langle v'_\alpha v'_\beta \rangle = & -\langle \rho \rangle \left( \langle v'_\alpha v'_\gamma \rangle \frac{\partial \langle v_\beta \rangle}{\partial x_\gamma} + \langle v'_\beta v'_\gamma \rangle \frac{\partial \langle v_\alpha \rangle}{\partial x_\gamma} \right) + \langle p' \left( \frac{\partial v'_\alpha}{\partial x_\beta} + \frac{\partial v'_\beta}{\partial x_\alpha} \right) \rangle \\ & + \frac{\partial}{\partial x_\gamma} \left( \mu \frac{\partial \langle v'_\alpha v'_\beta \rangle}{\partial x_\gamma} - \langle \rho \rangle \langle v'_\alpha v'_\beta v'_\gamma \rangle - \delta_{\alpha\gamma} \langle p' v'_\beta \rangle - \delta_{\beta\gamma} \langle p' v'_\alpha \rangle \right) - \langle \rho \rangle \tilde{\epsilon}_{\alpha\beta} \end{aligned} \quad (86)$$

where the dissipation rate tensor is defined by (1). The split of the pressure correlation into pressure transport and pressure rate of strain correlations can be shown to be inappropriate near the wall. It follows from (19) that the Taylor series for the pressure gradient is given in terms of velocity derivatives at the wall up to second order. Hence, it is possible to represent the Taylor series for correlations involving the pressure gradient in terms of local velocity correlations at the wall up to second order. If the pressure correlations are split as in (86) the pressure fluctuation itself appears and the solution of the Poisson equation introduces the well known integral contributions.

The standard second order closures employ the equation for the trace of the dissipation rate tensor

$$\epsilon \equiv \frac{1}{2} \langle \epsilon_{\alpha\alpha} \rangle$$

which follows at once from (2)

$$(\partial_t + \langle v_\gamma \rangle \partial_\gamma) \epsilon = \partial_\gamma [\nu \partial_\gamma \epsilon - \langle v'_\gamma \epsilon \rangle + F_\gamma^p] + S^1 + S^2 + S^3 + S^5 - D \quad (87)$$

The source terms are defined as follows. There are two groups of secondary production terms

$$S^1 \equiv -\langle \epsilon_{\alpha\gamma} \rangle \partial_\gamma \langle v_\alpha \rangle \quad (88)$$

and

$$S^2 \equiv -\langle \nu \partial_\delta v'_\alpha \partial_\gamma v'_\alpha \rangle (\partial_\gamma \langle v_\delta \rangle + \partial_\delta \langle v_\gamma \rangle) \quad (89)$$

The primary production term is given by

$$S^3 \equiv -\langle \epsilon_{\alpha\gamma} \partial_\gamma v'_\alpha \rangle \quad (90)$$

and the pressure correlations can be given in the form

$$S^4 \equiv -\frac{2\nu}{\rho} \langle \partial_{\alpha\gamma}^2 p' \partial_{\gamma} v'_{\alpha} \rangle \quad (91)$$

It was shown in chapter 1.2.1 that the pressure correlations can be represented as the sum of the divergence of a flux and a redistributive source which has zero trace. Hence we get

$$S^4 = \partial_{\delta} F_{\delta}^p \quad (92)$$

where the flux is defined by

$$F_{\delta}^p \equiv \frac{2\nu}{\rho} \epsilon_{\omega\delta\eta} \epsilon_{\omega\gamma\alpha} \langle \partial_{\gamma} p' \partial_{\eta} v'_{\alpha} \rangle \quad (93)$$

or

$$F_{\delta}^p = -\frac{2\nu}{\rho} \langle \partial_{\gamma} p' \partial_{\gamma} v'_{\delta} \rangle \quad (94)$$

The near wall production of dissipation rate is proportional to the curvature of the mean velocity profile

$$S^5 \equiv -2\nu \langle v'_{\delta} \partial_{\gamma} v'_{\alpha} \rangle \partial_{\gamma\delta}^2 \langle v_{\alpha} \rangle \quad (95)$$

and the viscous destruction of dissipation rate is given by

$$D \equiv 2\nu^2 \langle \partial_{\delta\gamma}^2 v'_{\alpha} \partial_{\delta\gamma}^2 v'_{\alpha} \rangle \quad (96)$$

The properties of the source terms have been established in chapter 1. and we observe that none of the source terms is closed in contrast to the equations for the dissipation rate tensor.

## 2.2 Standard second order closure model.

The development of the full second order closure was based in a systematic way on existing closure schemes. The standard second order closure using the trace of the dissipation rate tensor set up by Vandromme et al. (1983) was an important stepping stone and can be applied to test closure models for the various processes governing the dissipation rate tensor. The model is given by

$$\begin{aligned} \langle \rho \rangle \left( \frac{\partial}{\partial t} + \langle v_{\gamma} \rangle \frac{\partial}{\partial x_{\gamma}} \right) \langle v'_{\alpha} v'_{\beta} \rangle &= -\langle \rho \rangle \left( \langle v'_{\alpha} v'_{\gamma} \rangle \frac{\partial \langle v_{\beta} \rangle}{\partial x_{\gamma}} + \langle v'_{\beta} v'_{\gamma} \rangle \frac{\partial \langle v_{\alpha} \rangle}{\partial x_{\gamma}} \right) \\ &+ \Phi_{\alpha\beta} + \frac{\partial}{\partial x_{\gamma}} \left\{ \mu \frac{\partial \langle v'_{\alpha} v'_{\beta} \rangle}{\partial x_{\gamma}} - F_{\alpha\beta\gamma} \right\} - \langle \rho \rangle \langle \epsilon_{\alpha\beta} \rangle \end{aligned} \quad (97)$$

The pressure rate of strain correlations are modelled by

$$\Phi_{\alpha\beta} \doteq -c_1 f_p(R_e) \langle \rho \rangle \frac{\epsilon}{k} (\langle v'_{\alpha} v'_{\beta} \rangle - \frac{2}{3} \delta_{\alpha\beta} k) - f_p(R_e) \langle \rho \rangle \left\{ \frac{c_2 + 8}{11} (P_{\alpha\beta} - \frac{2}{3} \delta_{\alpha\beta} P) \right.$$

$$-\frac{8c_2-2}{11}(D_{\alpha\beta}-\frac{2}{3}\delta_{\alpha\beta}P)-\frac{30c_2-2}{55}k(\frac{\partial\langle v_\beta\rangle}{\partial x_\alpha}+\frac{\partial\langle v_\alpha\rangle}{\partial x_\beta})+\Phi_{\alpha\beta}^w\} \quad (98)$$

as the sum of return to isotropy, fast response and wall contributions, where

$$P_{\alpha\beta} \equiv -\langle v'_\alpha v'_\gamma \rangle \frac{\partial\langle v_\beta \rangle}{\partial x_\gamma} - \langle v'_\beta v'_\gamma \rangle \frac{\partial\langle v_\alpha \rangle}{\partial x_\gamma} \quad (99)$$

and  $P \equiv 1/2 P_{\alpha\alpha}$  and

$$D_{\alpha\beta} \equiv -\langle v'_\alpha v'_\gamma \rangle \frac{\partial\langle v_\gamma \rangle}{\partial x_\beta} - \langle v'_\beta v'_\gamma \rangle \frac{\partial\langle v_\gamma \rangle}{\partial x_\alpha} \quad (100)$$

The wall contribution is given by

$$\begin{aligned} \Phi_{\alpha\beta}^w \approx & \langle \rho \rangle \{ c'_1 \frac{\epsilon}{k} (\langle v'_\alpha v'_\beta \rangle - \frac{2}{3} \delta_{\alpha\beta} k) + c'_2 (P_{\alpha\beta} - D_{\alpha\beta}) + c_5 k (\frac{\partial\langle v_\alpha \rangle}{\partial x_\beta} \\ & + \frac{\partial\langle v_\beta \rangle}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \frac{\partial\langle v_\gamma \rangle}{\partial x_\gamma}) \} \frac{k^{1.5}}{\epsilon y} \end{aligned} \quad (101)$$

The dissipation rate tensor must be modelled in terms of its trace and the anisotropy of the stress tensor

$$\langle \epsilon_{\alpha\beta} \rangle \approx \frac{\epsilon}{k} (f_s \langle v'_\alpha v'_\beta \rangle + (1-f_s) \frac{2}{3} \delta_{\alpha\beta} k) \quad (102)$$

where  $f_s$  depends on the local Reynolds number. The turbulent flux is modelled by

$$F_{\alpha\beta\gamma} \approx -c_{st} \langle \rho \rangle \frac{k}{\epsilon} \langle v'_\gamma v'_\delta \rangle \frac{\partial}{\partial x_\delta} \langle v'_\alpha v'_\beta \rangle \quad (104)$$

The constants are given by  $c_{st} = 0.25$ ,  $c_1 = 1.5$ ,  $c_2 = 0.4$ ,  $c'_1 = 0.1597$ ,  $c'_2 = 0.0133$ ,  $c_5 = 0.0041$ . The low Reynolds number functions are given by

$$f_s(R_e) = \exp\left(\frac{-2}{1+R_e/30}\right) \quad (104)$$

and

$$f_p(R_e) = \tanh(R_e/50) \quad (105)$$

The equation for the trace of the dissipation rate tensor requires closure for all relevant processes on the right hand side. Following Vandromme et al. (1983) the model

$$\begin{aligned} (\frac{\partial}{\partial t} + \langle v_\gamma \rangle \frac{\partial}{\partial x_\gamma}) \epsilon = & \frac{\partial}{\partial x_\gamma} [\nu \frac{\partial}{\partial x_\gamma} \epsilon + c_\epsilon \frac{k}{\epsilon} \langle v'_\alpha v'_\gamma \rangle \frac{\partial \epsilon}{\partial x_\alpha}] - c_{\epsilon 1} \langle \rho \rangle f_1(R_e) f_\nu(R_e, R_y) \frac{\epsilon}{k} \langle v'_\alpha v'_\beta \rangle \frac{\partial \langle v_\beta \rangle}{\partial x_\alpha} \\ & - c_{\epsilon 2} \langle \rho \rangle f_2(R_e) \frac{\epsilon}{k} [\epsilon - 2\nu \left( \frac{\partial \sqrt{k}}{\partial x_2} \right)^2] + 2c_\epsilon f_\nu \mu \langle v'_\gamma v'_\delta \rangle \frac{\epsilon}{k} \frac{\partial^2 \langle v_\alpha \rangle}{\partial x_\gamma \partial x_\delta} \frac{\partial^2 \langle v_\alpha \rangle}{\partial x_\gamma \partial x_\delta} \end{aligned} \quad (106)$$



The low Reynolds number functions are given by

$$f_\nu = [1 - \exp(-f_{\nu 1} R_y)]^2 (1 + \frac{f_{\nu 2}}{R_e}) \quad (107)$$

and

$$f_1 = 1 + \frac{1}{8(f_\nu + 10^{-10})^3} \quad (108)$$

$$f_2 = 1 - 0.22 \exp(-\frac{R_e^2}{6}) \quad (109)$$

where

$$R_y \equiv \frac{\sqrt{k}y}{\nu} \quad (110)$$

and  $f_{\nu 1} = 0.0165$ ,  $f_{\nu 2} = 20.5$ . The performance of this closure model was tested in flat plate boundary layers. The results are contained in fig.1 to fig.8. The mean velocity in fig.1 is in good agreement with the law of the wall. The turbulent Reynolds number and the damping function  $f_s(R_e)$  in fig.2 and fig.3 prove that the boundary layer is fully developed. The Reynolds stress components in fig.4 to fig.8 show the expected distributions with maximal values in reasonable agreement with the experiments.

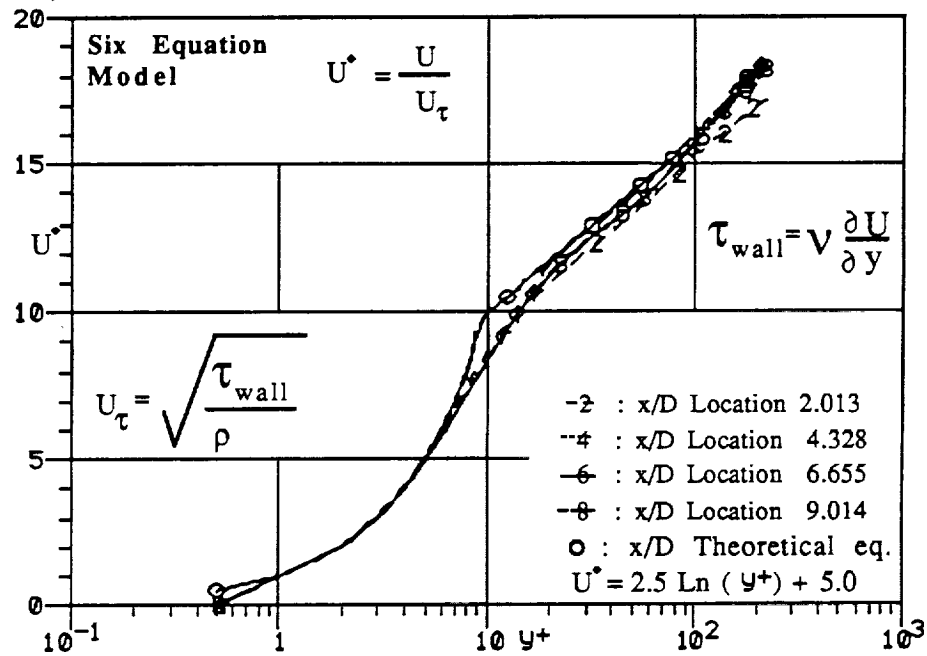


Figure 1 Variation of normalized mean velocity with wall distance in wall units

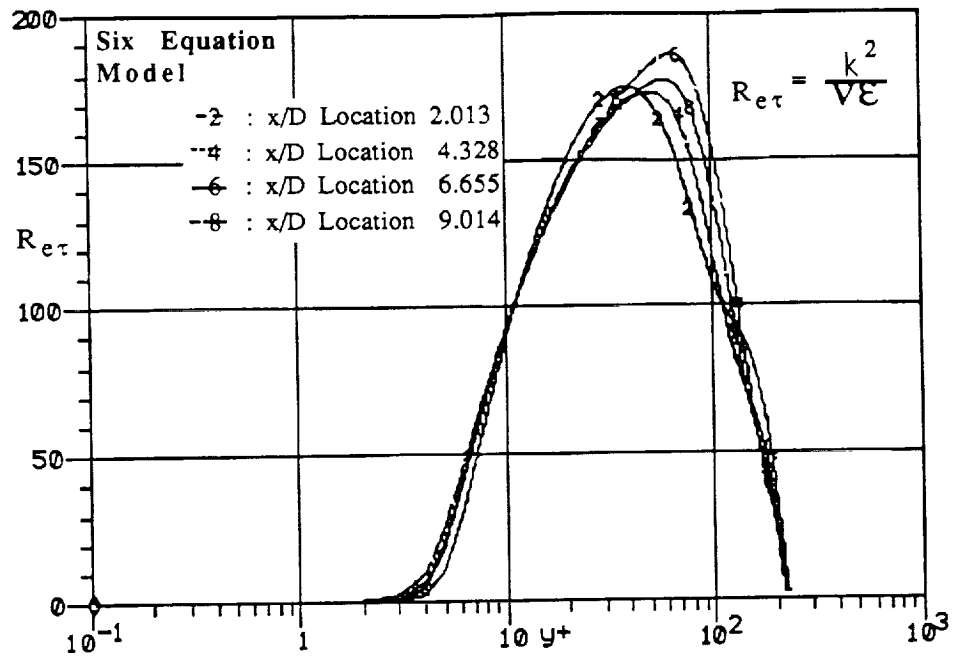


Figure 2 Variation of turbulent Reynolds number( $R_{e\tau}$ ) vs. normal distance from wall in wall units

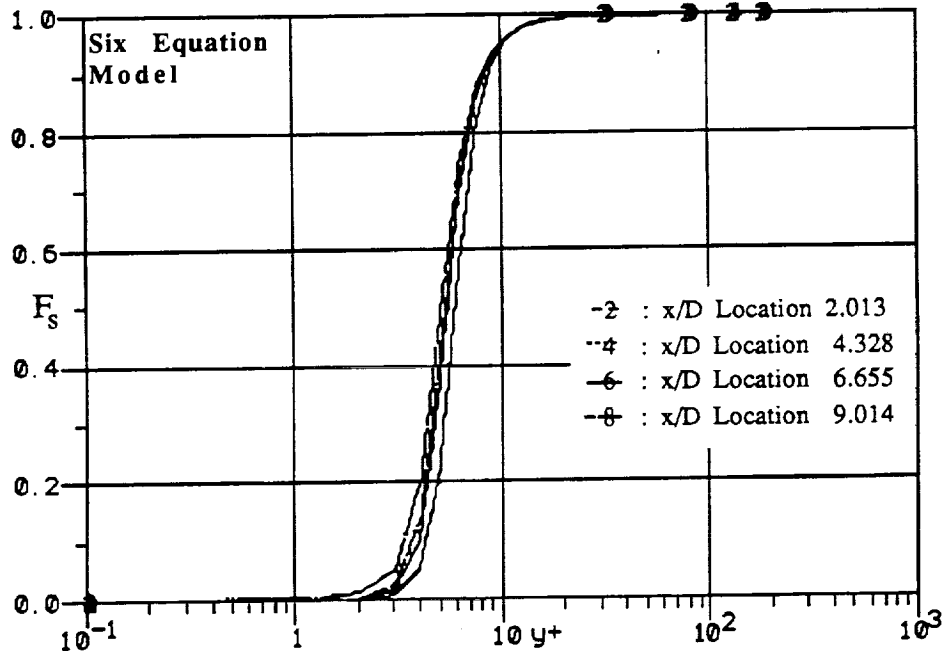


Figure 3 Damping function ( $F_s$ ) for dissipation rate term in Reynolds stress equation vs.  $y^+$

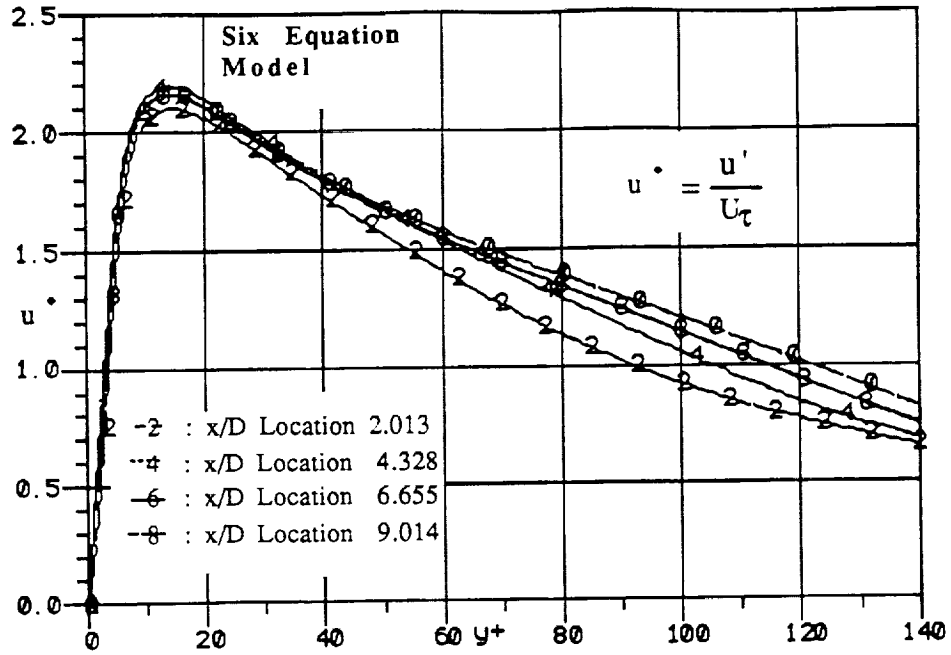


Figure 4 Variation of normalized fluctuating velocity  $u'$  with wall distance in wall units

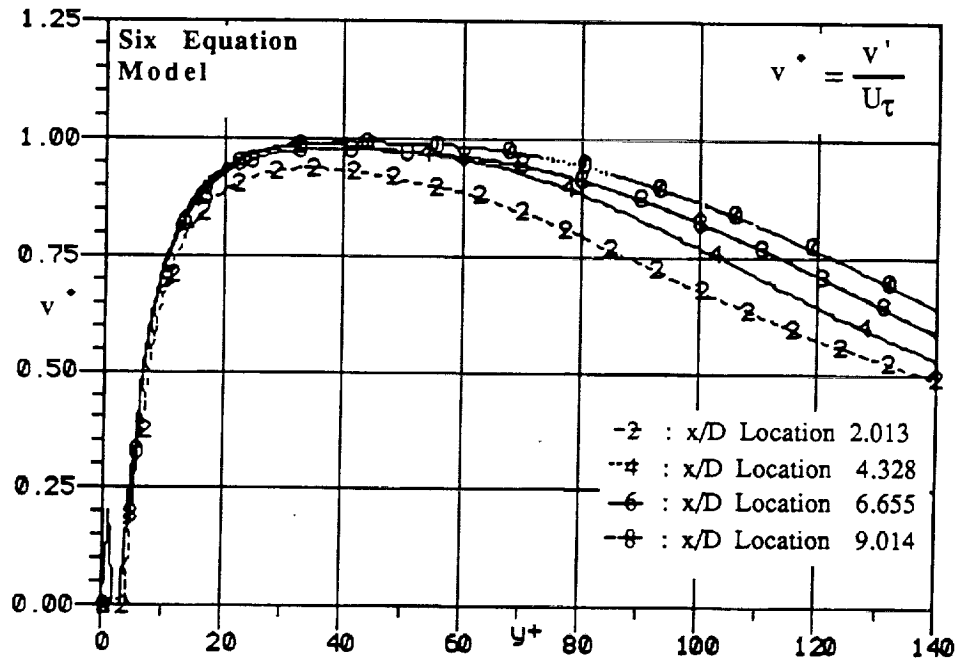


Figure 5 Variation of normalized fluctuation velocity  $v'$  with wall distance in wall units

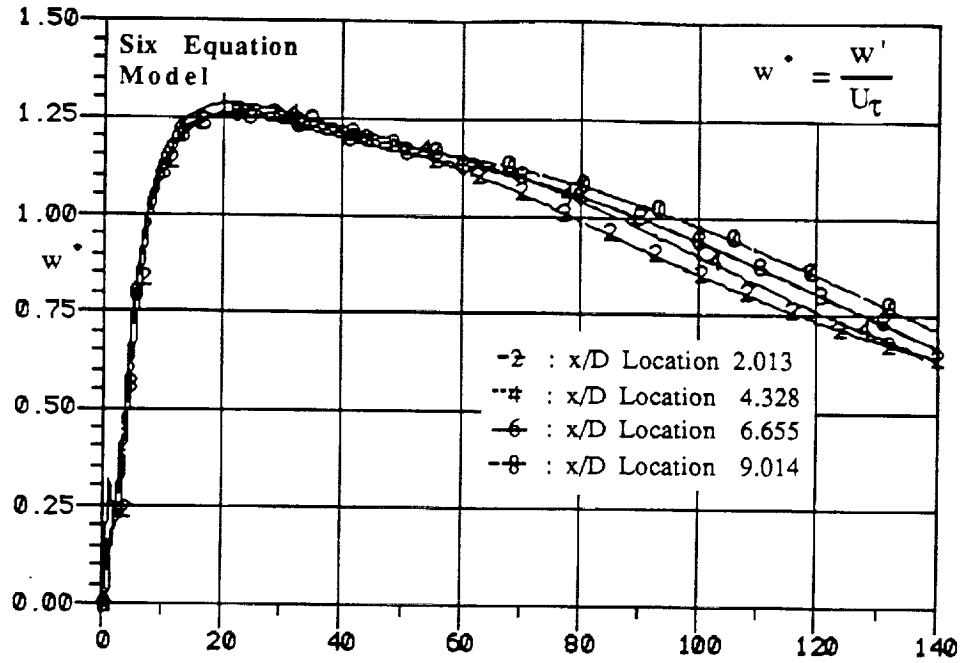


Figure 6 Variation of normalized fluctuating velocity  $w'$  with wall distance in wall units

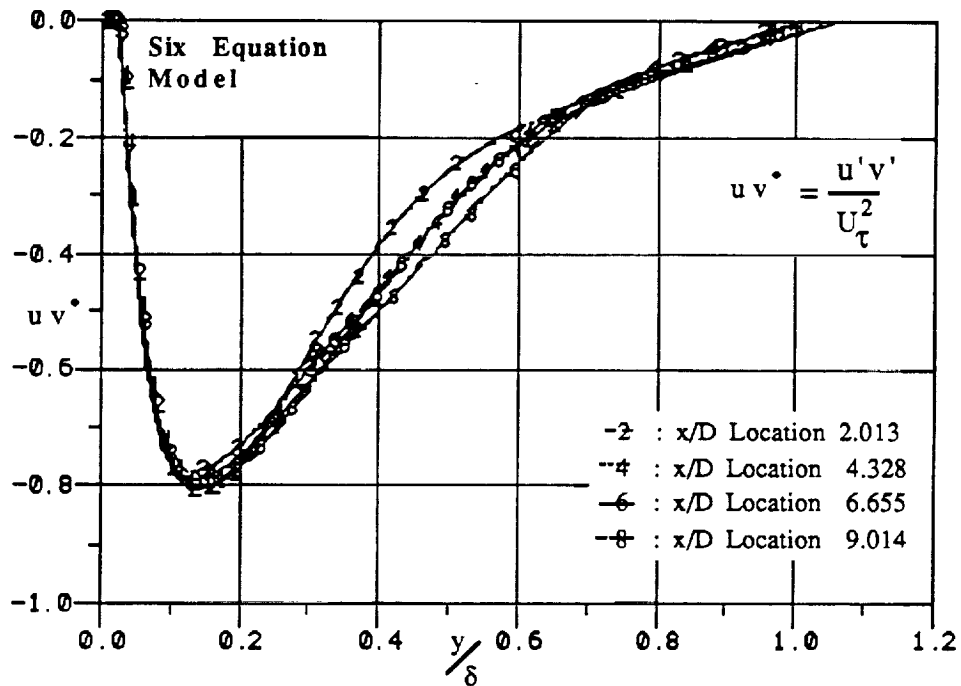


Figure 7a Variation of normalized shear stress  $u'v'$  with wall distance in boundary layer thickness

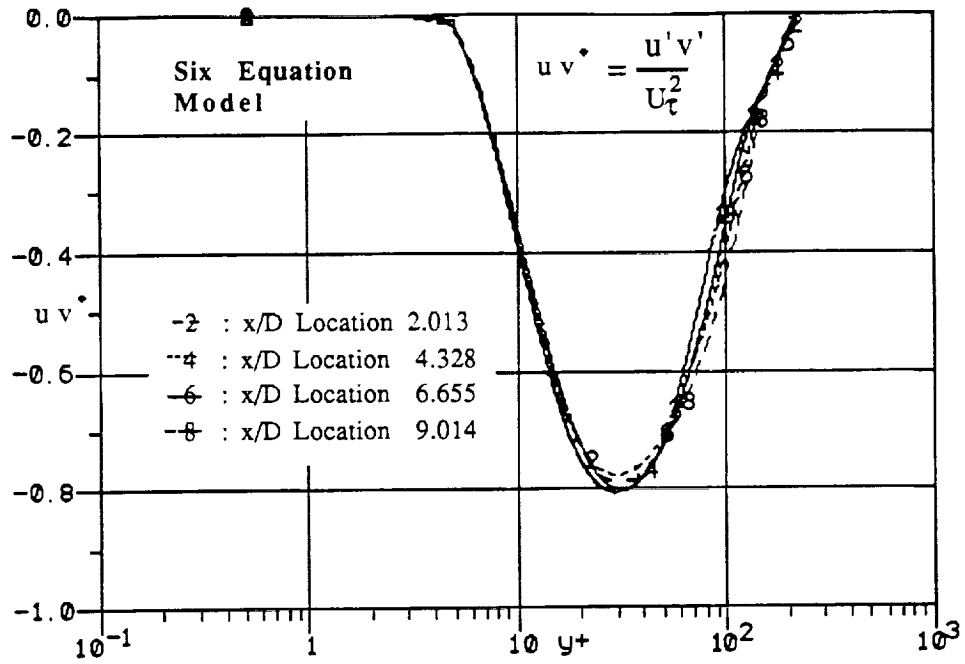


Figure 7b Variation of normalized sheer stress  $u'v'$  with wall distance in wall units

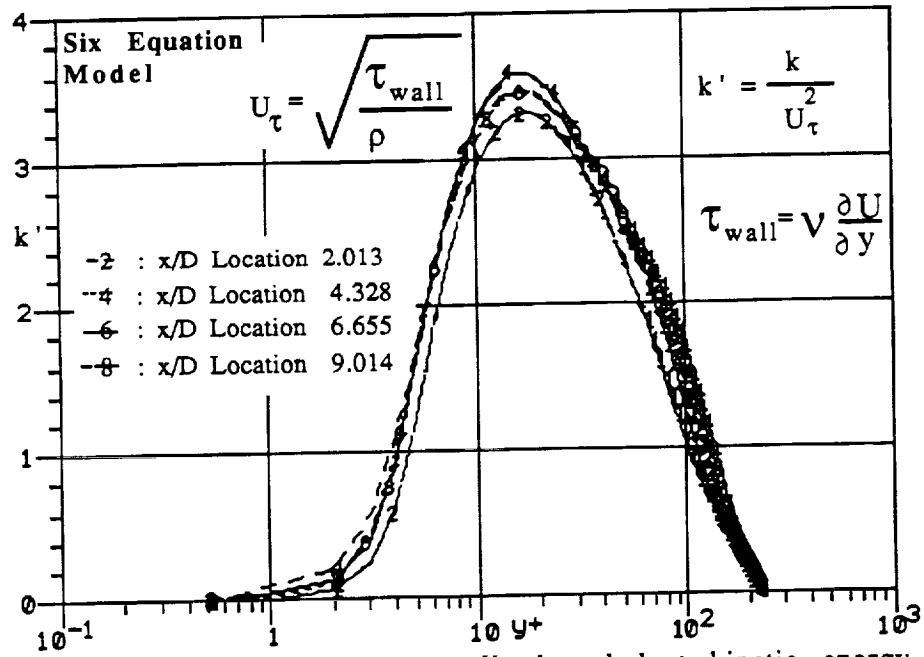


Figure 8 Variation of normalized turbulent kinetic energy with wall distance in wall units.

### 2.3 Full second order closure model.

The development of the full second order closure was prepared in the first chapter which contains the properties of all transport equations for second order moments derived from series expansions with respect to wall distance. The results obtained with series expansions are called growth laws since they describe in first or higher order the growth of correlations with wall distance. The challenge is now to construct a closure model that satisfies all tensorial and realizability conditions and the growth laws.

#### 2.3.1 Closure model for the stress equations.

The closure model for the stress equations follows closely the model developed by Vandromme et al. (1983). The only difference is that the model (102) for the components of the dissipation rate tensor is not used. The stress equations appear then according to chapter 2.2 as

$$\begin{aligned}
\langle \rho \rangle \left( \frac{\partial}{\partial t} + \langle v_\gamma \rangle \frac{\partial}{\partial x_\gamma} \right) \langle v'_\alpha v'_\beta \rangle = & -\langle \rho \rangle \left( \langle v'_\alpha v'_\gamma \rangle \frac{\partial \langle v_\beta \rangle}{\partial x_\gamma} + \langle v'_\beta v'_\gamma \rangle \frac{\partial \langle v_\alpha \rangle}{\partial x_\gamma} \right) \\
& - c_1 f_p(R_e) \langle \rho \rangle \frac{\epsilon}{k} \left( \langle v'_\alpha v'_\beta \rangle - \frac{2}{3} \delta_{\alpha\beta} k \right) - f_p(R_e) \langle \rho \rangle \left\{ \frac{c_2 + 8}{11} (P_{\alpha\beta} - \frac{2}{3} \delta_{\alpha\beta} P) - \frac{8c_2 - 2}{11} (D_{\alpha\beta} - \frac{2}{3} \delta_{\alpha\beta} P) \right. \\
& \quad \left. - \frac{30c_2 - 2}{55} k \left( \frac{\partial \langle v_\beta \rangle}{\partial x_\alpha} + \frac{\partial \langle v_\alpha \rangle}{\partial x_\beta} \right) + \Phi_{\alpha\beta}^w \right\} \\
& + \frac{\partial}{\partial x_\gamma} \left\{ \mu \frac{\partial \langle v'_\alpha v'_\beta \rangle}{\partial x_\gamma} + c_{st} \langle \rho \rangle \frac{k}{\epsilon} \langle v'_\gamma v'_\delta \rangle \frac{\partial}{\partial x_\delta} \langle v'_\alpha v'_\beta \rangle \right\} - \langle \rho \rangle \langle \epsilon_{\alpha\beta} \rangle
\end{aligned} \tag{111}$$

where  $P_{\alpha\beta}$  and  $D_{\alpha\beta}$  are defined in (99) and (100) respectively and the wall contribution to the pressure correlations is given by (101). The low Reynolds number functions are all established in chapter 2.2.

#### 2.3.2 Closure model for the dissipation rate equations.

The transport equation for the dissipation rate tensor

$$\left( \frac{\partial}{\partial t} + \langle v_\gamma \rangle \frac{\partial}{\partial x_\gamma} \right) \langle \epsilon_{\alpha\beta} \rangle = \frac{\partial F_{\alpha\beta\gamma}}{\partial x_\gamma} + S_{\alpha\beta}^1 + S_{\alpha\beta}^2 + S_{\alpha\beta}^3 + S_{\alpha\beta}^4 + S_{\alpha\beta}^5 - D_{\alpha\beta} \tag{112}$$

requires closure for the turbulent flux contained in the total flux  $F_{\alpha\beta\gamma}$  and the source terms  $S_{\alpha\beta}^2$ ,  $S_{\alpha\beta}^3$ ,  $S_{\alpha\beta}^4$ ,  $S_{\alpha\beta}^5$  and  $D_{\alpha\beta}$ .

Model for the turbulent flux: This model follows the suggestion of Vandromme et al. (1983) for the turbulent flux of the trace of the dissipation rate tensor

$$F_{\alpha\beta\gamma} \cong \nu \frac{\partial \langle \epsilon_{\alpha\beta} \rangle}{\partial x_\gamma} + c_\epsilon \langle \rho \rangle \frac{k}{n_\zeta n_\eta \epsilon_{\zeta\eta}} \langle v'_\gamma v'_\delta \rangle \frac{\partial \langle \epsilon_{\alpha\beta} \rangle}{\partial x_\delta} \quad (113)$$

The only modification is the use of the time scale (69) to improve the behaviour near the wall. The value for the constant  $c_\epsilon = 0.1$  is consistent with the six-equation model. A more sophisticated model was developed in chapter 1.3.2.

Model for the source  $S_{\alpha\beta}^2$ : The secondary production can be neglected near the wall according to chapter 1.2.2.4.

Model for the source  $S_{\alpha\beta}^3$ : The primary production is not of leading order near the wall but varies like the closed production term  $S_{\alpha\beta}^1$  in the near wall region according to ch.1.2.2.5. The present model utilizes this property and the analogy to the stress transport exploited in the equation for the trace (106), where the high Re-number part of the model for the difference between the leading terms having the character of a production term is modelled proportional to the production of kinetic energy

$$S_{\alpha\beta}^3 \cong -c_{\epsilon 3} \langle \rho \rangle \left[ \langle \epsilon_{\beta\gamma} \rangle \frac{\partial \langle v_\alpha \rangle}{\partial x_\gamma} + \epsilon_{\alpha\gamma} \frac{\partial \langle v_\beta \rangle}{\partial x_\gamma} \right] - c_{\epsilon 1} f_\nu(R_{et}) \langle \rho \rangle \frac{\epsilon}{k} \left[ \langle v'_\beta v'_\gamma \rangle \frac{\partial \langle v_\alpha \rangle}{\partial x_\gamma} + \langle v'_\alpha v'_\gamma \rangle \frac{\partial \langle v_\beta \rangle}{\partial x_\gamma} \right] \quad (114)$$

The constants are given by  $c_{\epsilon 1} = 1.45$  and  $c_{\epsilon 3} = 1.0$ .

Model for the source term  $S_{\alpha\beta}^4$ : The pressure correlations were shown to be of leading order near the wall for all dissipation rate components except  $\epsilon_{22}$  (see ch.1.2.2.9). They can be split into diffusive and redistributive source terms (ch.1.2.1). The diffusive part is assumed to be represented by the closure for the turbulent flux (113). The model for the redistributive source is analogous to the return to isotropy model for the stress transport equations. It is given by

$$S_{\alpha\beta}^4 \cong -c_{\epsilon 4} \langle \rho \rangle \frac{\epsilon}{k} (\langle \epsilon_{\alpha\beta} \rangle - \frac{2}{3} \delta_{\alpha\beta} \epsilon) \quad (115)$$

with  $c_{\epsilon 4} = 12.5$ .

Model for the destructive term  $D_{\alpha\beta}$ : It was shown in chapter 1.3.4 that the high Re-number model for the difference of primary production and viscous destruction becomes singular as the wall is approached. Furthermore, the growth law for the viscous destruction term  $D_{\alpha\beta}$  implies that a model similar to the high Re-number case

$$D_{\alpha\beta} \cong c_{\epsilon 2} \langle \rho \rangle \frac{\langle \epsilon_{\alpha\beta} \rangle}{\tau}$$

with a scalar time scale is impossible. It follows that a tensorial time scale must be constructed to conform with the growth law. The present closure is a composite expression containing the high and the low Re-number models

$$D_{\alpha\beta} \cong c_{\epsilon 21} \langle \rho \rangle (1 - f_s(R_{et})) \left( \frac{k}{n_\delta n_\zeta \langle \epsilon_{\delta\zeta} \rangle} \right)^2 n_\eta n_\phi \frac{\partial \langle \epsilon_{\alpha\gamma} \rangle}{\partial x_\eta} \frac{\partial \langle \epsilon_{\beta\gamma} \rangle}{\partial x_\phi}$$

$$+c_{\epsilon 22}\langle\rho\rangle n_{\alpha}n_{\beta}n_{\gamma}n_{\delta}(1-f_s(R_{et}))\frac{\epsilon}{k}\langle\epsilon_{\gamma\delta}\rangle + c_{\epsilon 23}\langle\rho\rangle f_s(R_{et})\frac{n_{\gamma}n_{\delta}\langle\epsilon_{\gamma\delta}\rangle}{k}\langle\epsilon_{\alpha\beta}\rangle \quad (116)$$

where the preliminary values for the constants are  $c_{\epsilon 21} = 0.32 \cdot 10^{-4}$ ,  $c_{\epsilon 22} = 26.25$ , and  $c_{\epsilon 23} = 12.5$ . The low Re-number functions are set up as follows:

$$f_{\nu}(R_{et}) = (1 - \exp[-0.0165R_y])^2(1 - \frac{20.5}{R_{et}}) \quad (117)$$

where

$$R_y \equiv \frac{\sqrt{k}y}{\nu}$$

denotes the dimensionless wall distance and

$$f_s(R_{et}) = \tanh(0.004R_{et}) \quad (118)$$

The turbulent Reynolds number is defined by

$$R_{et} = \frac{k^2}{\epsilon\nu}$$

### 2.3.3 Preliminary results for the complete second order closure.

The system of nine parabolic differential equations was tested in reduced form by prescribing the profiles for mean velocity and the Reynolds stress components which were obtained with the six equation model discussed in chapter 2.2. The numerical solution for the remaining equations for the dissipation rate tensor was carried out and convergence was achieved after a few hundred steps. The results are presented in fig.9 to fig.16. The figures contain also as broken line the dissipation rate components deduced from the local relation suggested by Launder and Reynolds (1983) and modified by Lai and So (1990)

$$\begin{aligned} \langle\epsilon_{\alpha\beta}\rangle &\cong \frac{2}{3}(1 - f_w(R_{et}))\delta_{\alpha\beta}\epsilon \\ &+ f_w(R_{et})\frac{\epsilon}{k} \frac{\langle v'_{\alpha}v'_{\beta}\rangle + \langle v'_{\alpha}v'_{\gamma}\rangle n_{\gamma}n_{\beta} + \langle v'_{\beta}v'_{\gamma}\rangle n_{\gamma}n_{\alpha} + n_{\alpha}n_{\beta}n_{\gamma}n_{\delta}\langle v'_{\gamma}v'_{\delta}\rangle}{1 + \frac{3}{2k}n_{\gamma}n_{\delta}\langle v'_{\gamma}v'_{\delta}\rangle} \end{aligned} \quad (119)$$

where

$$f_w(R_{et}) = \exp[-(\frac{R_{et}}{150})^2]$$

The dissipation rates in the figures are normalized with the wall variables  $\nu$  and  $u_{\tau}$

$$\epsilon_{\alpha\beta}^+ \equiv \frac{\langle\epsilon_{\alpha\beta}\rangle\nu}{u_{\tau}^4}$$



with  $u_\tau = \sqrt{\tau_w/\rho}$  denoting the wall shear velocity. The symbols in the figures represent the results of the numerical experiments carried out by Mansour, Kim and Moin (1988).

The component  $\epsilon_{11}^+$  in fig.9 (logarithmic scale) and fig.10 (linear scale) shows a negative gradient near the wall which implies that the time scale

$$\tau^{-1} = \frac{\epsilon}{k} - 2\frac{\nu}{k} \left( \frac{\partial \sqrt{k}}{\partial y} \right)^2$$

would change sign and the destructive term would become a production term violating realizability. The prediction of  $\epsilon_{11}^+$  in the outer (fully turbulent) part of the flow field is too small and the near wall part appears too large compared to the direct simulation, but the profile shape is in good agreement with the numerical experiment. The component  $\epsilon_{22}^+$  in fig.11 and fig.12 reasonable agreement between the full second order closure and the direct simulation, but the local relation of Launder and Reynolds overpredicts the component by a factor of three. The prediction of the component  $\epsilon_{33}^+$  in fig.13 and fig.14 shows a similar behaviour as the component  $\epsilon_{11}^+$  in fig.9/10. The shear dissipation  $\epsilon_{12}^+$  in fig.15/16 shows overprediction by the full second order closure and underprediction by the local relation near the wall whereas the outer part is in good agreement with the direct simulation data.

#### 2.3.4 Conclusions.

The results presented lead to several conclusions. It is clear from the theoretical development that only the full second order closure offers the tools to construct the appropriate time scales in the near wall region of a turbulent boundary layer. The growth laws for the various correlations appearing in the stress and dissipation rate balances limit severely the model expressions and indicate that composite models for the high and the low Reynolds number regimes must be established. The model discussed in this chapter produces good results if velocity and stress components are held fixed and this indicates that the model expressions are consistent with the direct simulations. However, the stability of the closure needs to be investigated and this part may lead to modifications of the present version of the full second order closure. This part of the project is currently under way.

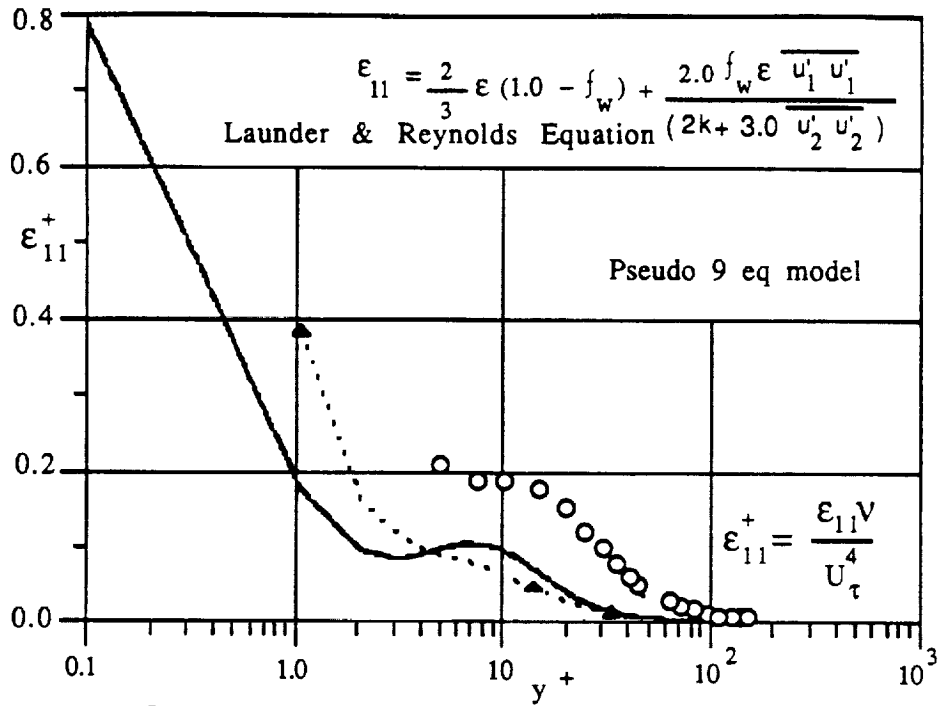


Figure 9 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

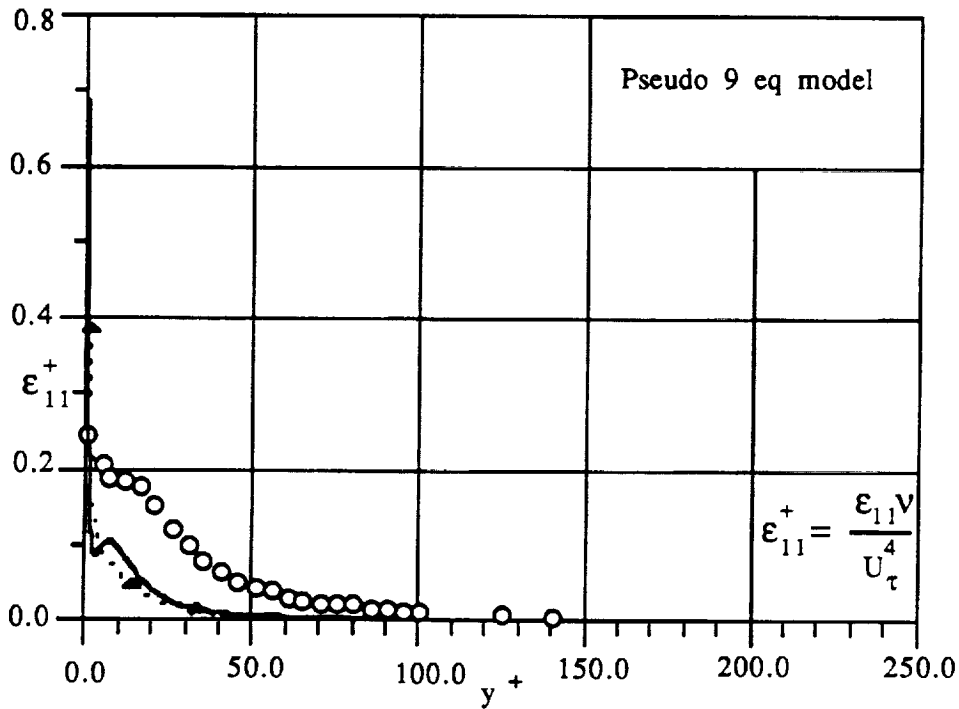


Figure 10 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

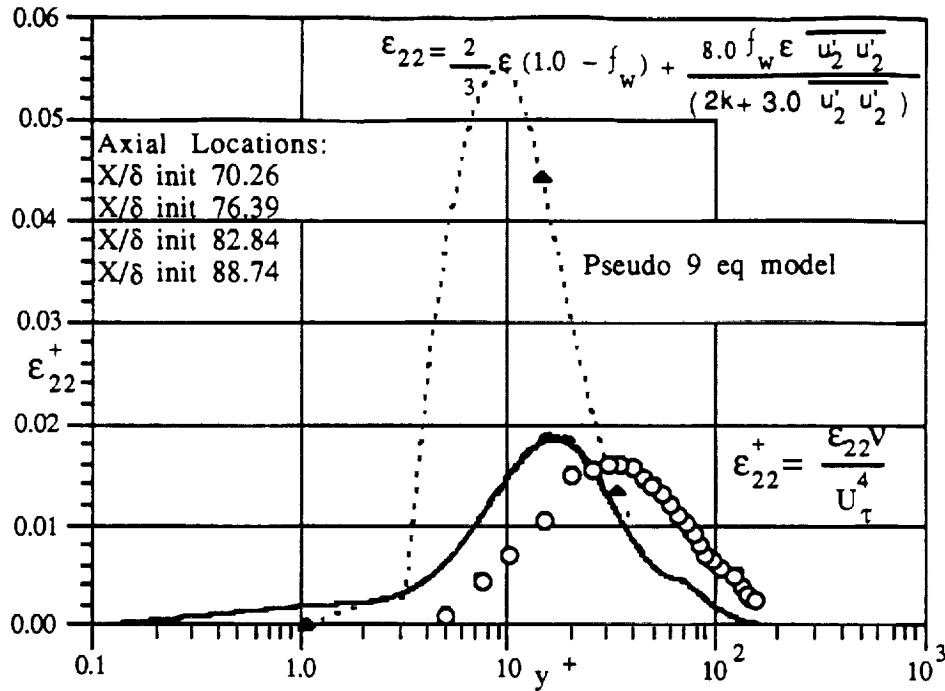


Figure 11 : Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — : Present Work.

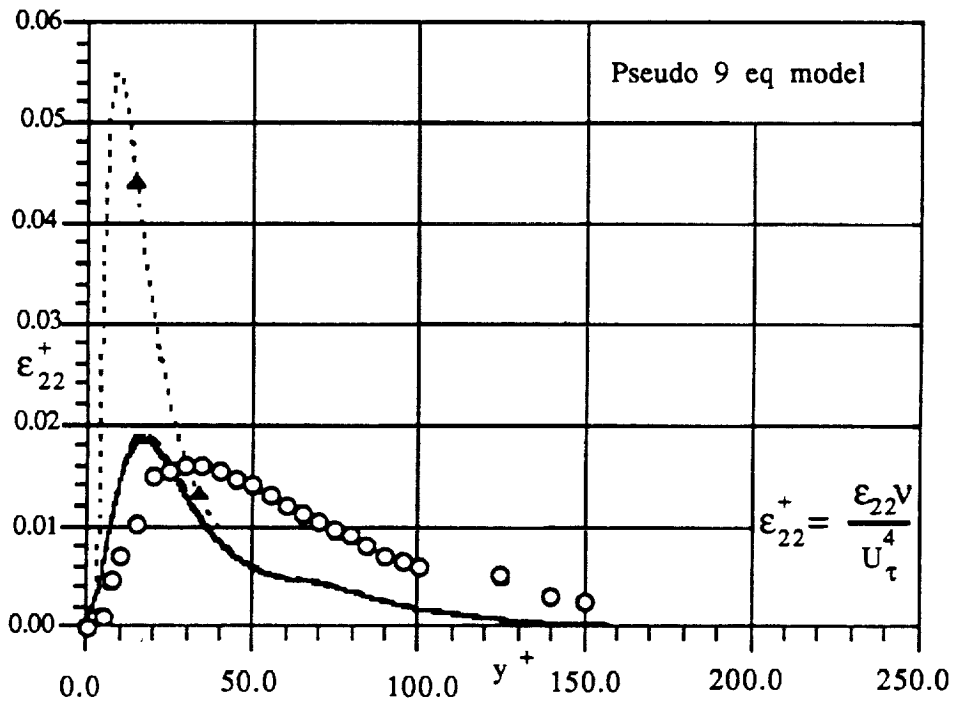


Figure 12 : Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — : Present Work.

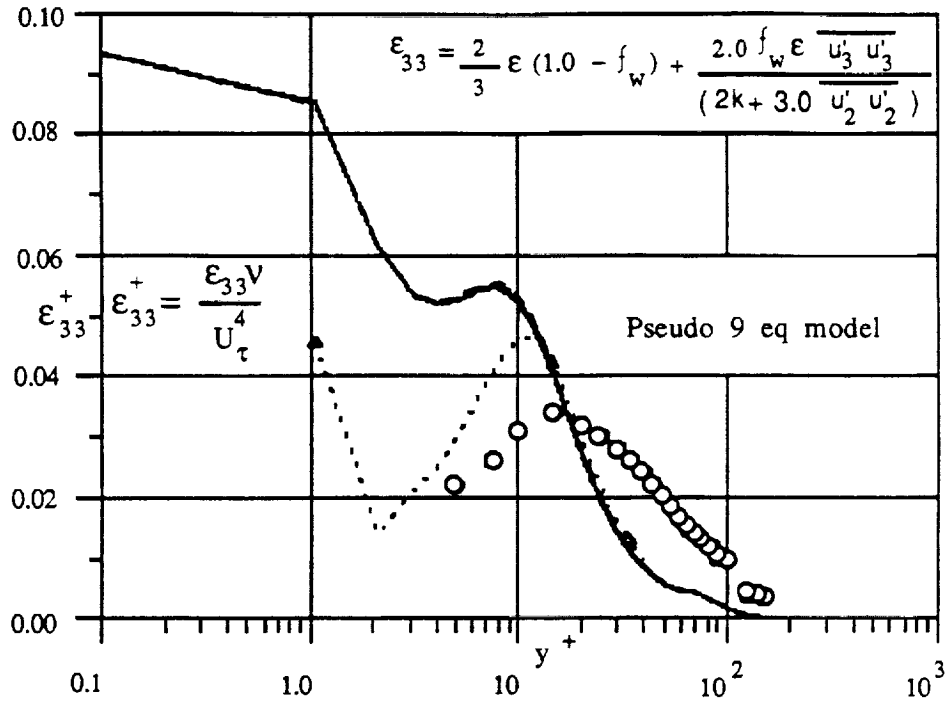


Figure 13 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

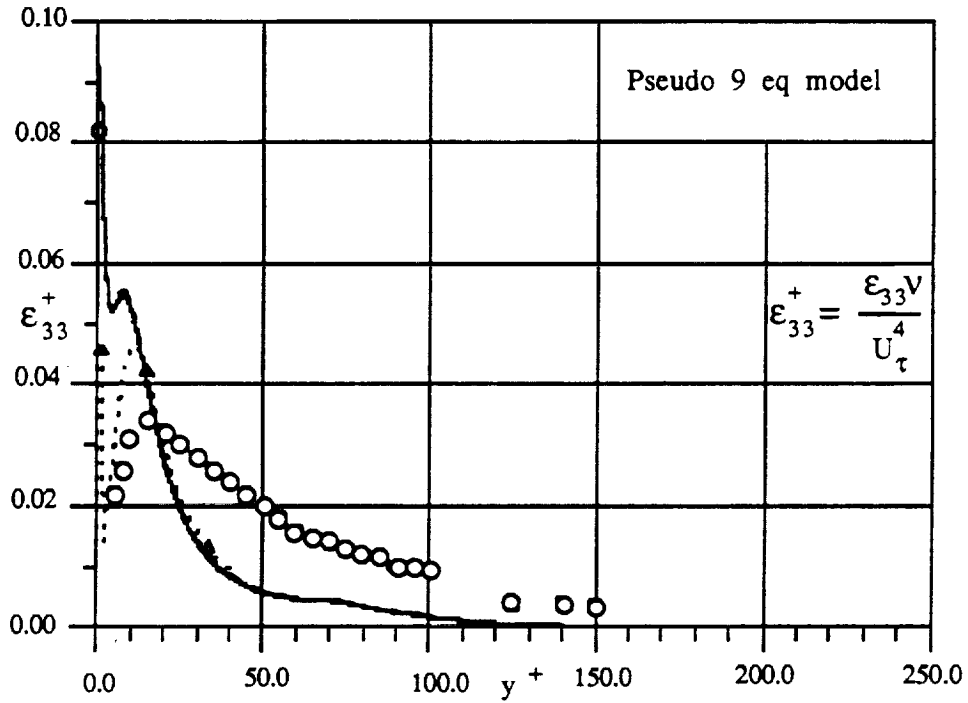


Figure 14 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

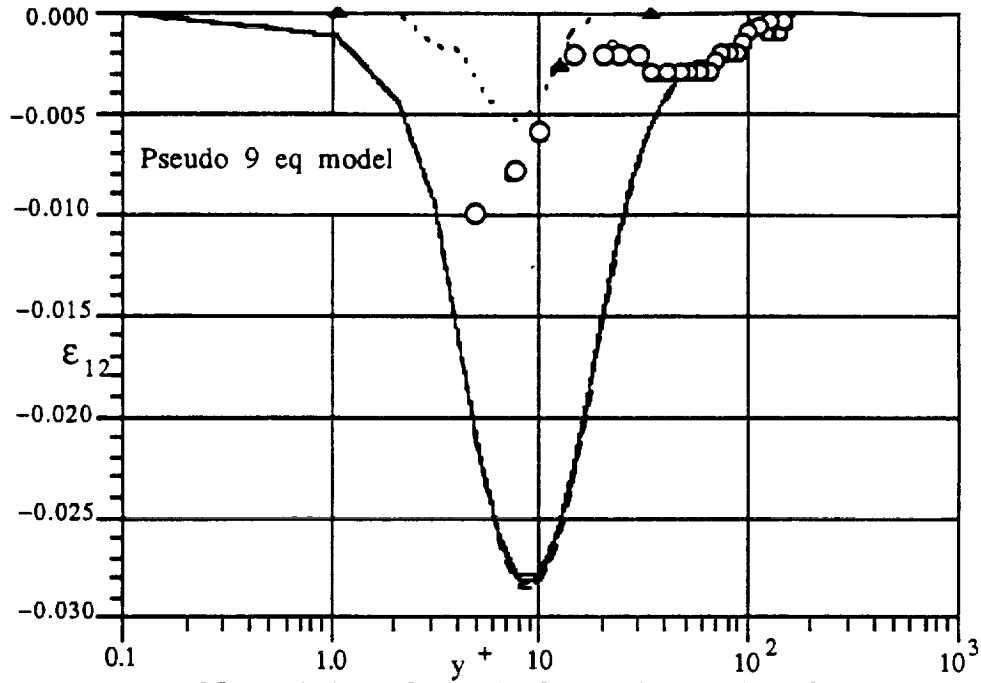


Figure 15 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

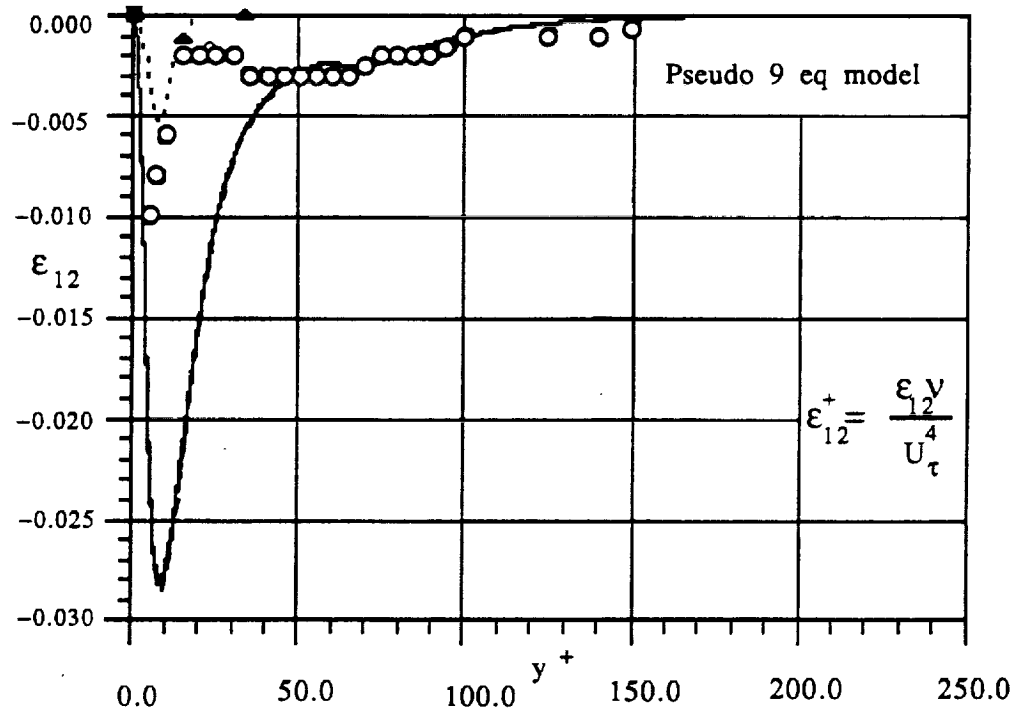


Figure 16 :Variation of Psuedo 9 equation model of normalized dissipation rated with wall distance. ○ : Results from Mansour, Kim, and Moin direct simulation results. ▲ Launder & Reynolds. — :Present Work.

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